

TITS GEOMETRY AND POSITIVE CURVATURE

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The interest in positively curved manifolds goes back to the beginning of Riemannian geometry or even to spherical and projective geometry. Likewise, the program of Tits to provide an axiomatic description of geometries whose automorphism group is a noncompact simple algebraic or Lie group goes back to projective geometry.

The presence of symmetries has played a significant role in the study of positively curved manifolds during the past two decades; see e.g. the surveys [Gr, Wi1, Zi]. Not only has this resulted in a number of classification type theorems, it has also lead to new insights about structural properties, see e.g. [VZ, Wi3], as well as to the discovery and construction of a new example [De, GVZ].

Unlike [GWZ], our work here is not motivated by the desire to find new examples. On the contrary, we wish to explore *rigidity properties* of special actions on positively curved manifolds whose linear counterparts by work of Dadok [Da], Cartan (see [He]), Tits [Ti1], and Burns and Spatzier [BSp] ultimately are described axiomatically via so-called *compact spherical buildings*.

The special actions we investigate are the so-called *polar actions*, i.e., isometric actions for which there is an (immersed) submanifold, a so-called *section*, that meets all orbits orthogonally. Such actions form a particularly simple, yet very rich and interesting class of manifolds and actions closely related to the transformation group itself. The concept goes back to isotropy representations of symmetric spaces. Also, as a special case, the adjoint action of a compact Lie group on itself is polar with section a maximal torus. Its extension to general manifolds was pioneered by Szenthe in [Sz] and independently by Palais and Terng in [PTe], and has recently been further developed in [GZ]. Since the action by the identity component of a polar action is itself polar, we assume throughout without further comments that our group is connected. An exceptional but important special case is that of cohomogeneity one actions and manifolds, i.e., actions with 1-dimensional orbit space.

The *exceptional case* of positively curved cohomogeneity one manifolds was studied in [GWZ]. Aside from the rank one symmetric spaces, this also includes infinite families of other manifolds, most of which are not homogeneous even up to homotopy. In contrast, our main result here is the following:

THEOREM A. *A polar action on a simply connected, compact, positively curved manifold of cohomogeneity at least two is equivariantly diffeomorphic to a polar action on a compact rank one symmetric space.*

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This is reminiscent of the situation for *isoparametric* submanifolds in euclidean spheres, where many isoparametric hypersurfaces are not homogeneous (see [OT, FKM]), whereas in higher codimensions by [Th] they are the orbits of linear polar actions if they are irreducible or equivalently the orbits of isotropy representations of compact symmetric spaces by [Da].

All polar actions on the simply connected, compact rank one symmetric spaces, i.e., the spheres and projective spaces, \mathbb{S}^n , \mathbb{CP}^n , \mathbb{HP}^n and \mathbb{CaP}^2 were classified in [Da] and [PTh]. In all cases but \mathbb{CaP}^2 they are either linear polar actions on a sphere or they descend from such actions to a projective space. By the work mentioned above by Dadok, Cartan, Tits, and Burns-Spatzier, the (maximal) irreducible polar linear actions are in 1-1 correspondence with irreducible *compact spherical buildings*. On \mathbb{CaP}^2 any polar action has either cohomogeneity one or two, and in the second case all but one have a fixed point. The latter is an action by $\mathrm{SU}(3) \times \mathrm{SU}(3)$ with orbit space a spherical triangle with angles $\pi/2$, $\pi/3$ and $\pi/4$. We refer to this as the *exceptional* (irreducible) action on \mathbb{CaP}^2 .

There are different steps and strategies involved in the proof of Theorem A. To guide the reader we provide a short discussion of the key results needed in the proof.

Our point of departure is the following description of sections and their (effective) stabilizer groups referred to as *polar groups* in [GZ] and Weyl groups in [Sz, PTe]:

THEOREM B. *The polar group of a simply connected positively curved polar manifold of cohomogeneity at least two is a Coxeter group or a \mathbb{Z}_2 quotient thereof. Moreover, the section with this action is equivariantly diffeomorphic to a sphere, respectively a real projective space with a linear action.*

This allows us to associate a (connected) *chamber system* $\mathcal{C}(M; \mathbf{G})$ (cf. [Ti2, Ro]) of type M (the Coxeter matrix of the associated Coxeter group), to any simply connected positively curved polar \mathbf{G} manifold M of cohomogeneity at least two. We point out that in this generality, the geometric realization of $\mathcal{C}(M; \mathbf{G})$ is not a simplicial complex, so not a *geometry* of type M in the sense of Tits. For this we prove:

THEOREM C. *Let M be a simply connected positively curved polar \mathbf{G} manifold without fixed points, and not (equivalent to) the exceptional action on \mathbb{CaP}^2 . Then the universal cover $\tilde{\mathcal{C}}(M; \mathbf{G})$ of $\mathcal{C}(M; \mathbf{G})$ is a spherical building.*

Moreover, the Hausdorff topology on compact subsets of M induces in a natural way a topology on $\tilde{\mathcal{C}}(M; \mathbf{G})$ for which we prove:

THEOREM D. *Whenever the universal cover $\tilde{\mathcal{C}}(M; \mathbf{G})$ of $\mathcal{C}(M; \mathbf{G})$ is a building, it is a compact spherical building.*

When the Coxeter diagram for M is connected, or more generally has no isolated nodes, the work of Burns and Spatzier [BSp] as extended by Grundhöfer, Kramer, Van Maldeghem and Weiss [GKMW] applies, and hence $\tilde{\mathcal{C}}(M; \mathbf{G})$ is the building of the sphere at infinity of a noncompact symmetric space \mathbf{U}/\mathbf{K} of non-positive curvature, and the action of \mathbf{K} on the sphere at infinity is the linear polar action whose chamber system is the building. In our case, the fundamental group π of the cover becomes a compact normal subgroup of $\tilde{\mathbf{G}} \subset \mathbf{K}$ acting freely

on the sphere with quotient our manifold with the action by $G = \tilde{G}/\pi$. Moreover, the actions by \tilde{G} and K on the sphere are orbit equivalent. This already proves our Theorem A up to equivariant homeomorphism in this case, and equivariant diffeomorphism follows, e.g., from the recognition theorem in [GZ].

In the remaining (reducible) cases (including the case of fixed points), where isolated nodes of the Coxeter diagram are present, the above mentioned extended Burns-Spatzier-Tits theory does not yield the desired result, and we use more direct geometric arguments. Although it is possible to prove Theorem A for all reducible actions, without appealing to Tits geometry our proof is in essence done by induction on the number of isolated nodes of the Coxeter diagram via a characterization of Hopf fibrations in our context.

We point out that the proof of Theorem C above has two distinct parts. For all chamber systems of type $M \neq C_3$ our constructions combined with the work of Tits gives the result. In the exceptional case of C_3 , corresponding to the orbit space being a spherical triangle with angles $\pi/2$, $\pi/3$ and $\pi/4$, the general theory breaks down. Instead, we first prove that in this case $\mathcal{C}(M; G)$ is simplicial, i.e., a C_3 geometry allowing us to use an axiomatic characterization of buildings of type C_3 due to Tits. This is carried out in most cases via reductions and the work in [GWZ]. In contrast to the general theory where universal covers of our chamber systems automatically are buildings by the work of Tits, in the case of C_3 we construct covers and prove that they are buildings. In one case this is bound to fail due to theorem D above. In fact, we conclude:

THEOREM E. *The universal cover $\tilde{\mathcal{C}}$ of the chamber system $\mathcal{C}(\mathbb{CP}^2, \mathrm{SU}(3) \mathrm{SU}(3))$ for the exceptional action on \mathbb{CP}^2 is a C_3 geometry which is not a building.*

The existence of such C_3 geometries is well known in the “real estate community” (see [Ne]), but this particular example which arises very naturally in our context does not seem to have been noticed until now (see also [Ly]).

We like to mention that a corresponding theory for polar actions on non-negatively curved manifolds is significantly more involved, even when dealing with geometrically irreducible cases, i.e., non product cases. In particular, the concept of an irreducible action is not as straightforward in this case, since the section is no longer a sphere. For example the polar T^2 action on \mathbb{CP}^2 with three fixed points induces a polar T^2 action on $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ with a metric of nonnegative curvature and flat Klein bottle as section (see [GZ]) that should be viewed as reducible. With the appropriate notion of reducibility we

CONJECTURE. *An irreducible polar action on a simply connected nonnegatively curved compact manifold is equivariantly diffeomorphic to a polar action on a symmetric space.*

Another interesting direction for future work is based on some of our work here that generalizes to curvature free settings. For example, much of the (topological) work about spherical buildings carries over to situations of polar actions where the section is only a homotopy sphere, or a homotopy projective space.

We have divided the paper into eight sections and one appendix. The first two sections are devoted to preliminaries and an analysis of sections culminating in Theorem B. The chamber

system associated with a polar action in positive curvature is investigated in Section 3. The point of departure here is that this chamber system is connected. The proof of this is based on a result about dual foliations due to Wilking [Wi3]. Unlike previous applications of buildings to geometry, what is essential for us is to use the local approach to buildings and their universal covers. We like to mention that this is the case also in independent simultaneous work by Lytchak [Ly] describing the structure of polar singular foliations of codimension at least three in symmetric spaces. We conclude Section 3 by proving Theorem C in all cases but C_3 geometry.

In Section 4 we equip the ingredients of Theorem C with natural topologies based on the classical Hausdorff topology on closed sets in a compact metric space. Our main result here is that with this topology the universal covers of our chamber systems are compact spherical buildings in the sense of Burns and Spatzier. This then in particular leads to a proof of Theorem A for all irreducible actions. To complete the proof of Theorem C, and hence Theorem A in all irreducible cases, it remains to deal with the case of C_3 .

Sections 5 and 6 are devoted to proving that the universal cover of all but one chamber system of type C_3 in our context, indeed is a building. This is done almost uniformly via reductions to C_2 geometries recognized in [GWZ]. In fact we show that our chamber system or a cover constructed thereof satisfies an axiom for C_3 buildings due to Tits, by showing that a suitable reduction is C_2 . It is surprising that the geometry over the real field, which is usually considered the most complicated case, in some sense is simpler because of the existence of a reduction to A_3 geometry (Section 5). The only case where the universal cover may not be a building is then recognized (cf. [GZ]) by comparing its isotropy group data with that of the exceptional example of the isometric action of $SU(3) \times SU(3)$ on $\mathbb{C}a\mathbb{P}^2$ (cf. [PTh]). We mention that Kramer and Lytchak [KL] recently completed the classification of all homogeneous *geometries* whose Coxeter matrix has no isolated nodes, including the important new case of C_3 geometry using methods different from ours. When combined with [Ly], this in particular extends the structure result of [Ly] to codimension two in the case of polar actions on symmetric spaces.

As mentioned above, the general theory for compact spherical buildings breaks down for reducible actions in general (the ones for whom the Coxeter diagram has isolated nodes). The proof of Theorem A for such actions is carried out in Sections 7 and 8. Although we use buildings here as well, more direct arguments using so-called primitivity proved in Section 3 are used throughout. As a key input, we provide in Section 7 a characterization of Hopf fibrations in our context which is of independent interest. This immediately yields Theorem A for the special case where fixed points are present. The reducible case where no fixed points are present is dealt with in Section 8. In both of these sections we appeal to an understanding of the structure of reducible polar representations (including exceptional ones [Be]). Due to an oversight in the literature, this is provided in the appendix.

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1. PRELIMINARIES

We will begin by giving a brief description of known facts for general polar manifolds (cf. e.g. [GZ] and [HPPT] for further information) with emphasis on simply connected ones. Under fairly mild restrictions that always hold in positive curvature (see Section 2), there is a general

so-called chamber system naturally associated with such actions. We will end the section with a description of such systems, and the special case of Coxeter systems.

Throughout G will be a compact connected Lie group acting isometrically on a connected compact Riemannian manifold M in a polar fashion. By definition there is a section Σ , i.e., an immersion $\sigma : \Sigma \rightarrow M$ of a connected manifold Σ , whose image intersects all G orbits orthogonally. Moreover, we demand that σ is a section without a subcover section. Obviously $g\sigma$ is a section for any $g \in G$, and $G\sigma(\Sigma) = M$. Clearly, Σ has the same dimension as the *orbit space* $M^* := M/G$, i.e., the *cohomogeneity* of the action, or the codimension of *principal* orbits, $G/H \subset M$. If not otherwise stated it is understood that $0 < \dim M^* < \dim M$. This eliminates general actions by discrete groups, and general transitive actions. In addition, we also assume that M is not a product where G acts trivially on one of the factors. In general, we will denote the image of a subset $X \subset M$ under the *orbit* map by $X^* \subset M^*$.

The following facts are simple and well known (cf. [Sz, PTe]):

- Any section is *totally geodesic*.
- The slice representation of any isotropy group $K \subset G$ is a *polar representation*.

Fix a section $\sigma : \Sigma \rightarrow M$ and a point $p \in \Sigma$ corresponding to a principal G orbit, i.e., $G\sigma(p)$ is a principal orbit with isotropy group $H = G_{\sigma(p)}$. The stabilizer subgroup $G_{\sigma(\Sigma)} \subset G$ of $\sigma(\Sigma)$ induces an action on Σ . Clearly, H is the kernel of that action, and we refer to $\Pi := G_{\sigma(\Sigma)} / H$ as the *polar group* associated to the section σ . Recall the following facts:

- For any $q \in \Sigma$, $\sigma_*(T_q\Sigma) \subset T_{\sigma(q)}M$ is a section of the polar representation, the slice representation of $G_{\sigma(q)}$, and the associated polar group is the isotropy group Π_q .
- Π is a discrete subgroup of $N(H)/H$ acting properly discontinuously on Σ and freely on Σ_o , the regular part of Σ .
- $M^* = \Sigma^* := \Sigma/\Pi$ is an orbifold.

In complete generality, the structure of M and its G action is encoded in the section Σ , the polar group Π and its actions on Σ and G/H , and the G isotropy groups along Σ (cf. [GZ]). Although in general, Π can be any group, typically singular orbits are present, and there is a non-trivial normal subgroup. $W \subset \Pi$ generated by *reflections*, r_i associated with maximal singular isotropy groups $K_i \subset G$ along Σ . Components of the corresponding strata in $M^* = \Sigma^*$ are called (open) boundary faces, and their closures simply *boundary faces*. We stress that here $r : \Sigma \rightarrow \Sigma$ is called a reflection if r has order two, and at least one component of the fixed point set has codimension 1. The codimension 1 components $\Lambda_r \subset \Sigma$ of the fixed point set Σ^r are referred to as the *mirror* of r . We will refer to W as the *reflection group* of Σ . A connected component c of the complement of all mirrors is called an (*open*) *chamber* of Σ . We denote the closure of an open chamber by $C = \bar{c}$ and refer to it simply as a *chamber*. Again, we stress that this kind of terminology is usually reserved to the situation where the complement of a mirror has two connected components interchanged by the reflection.

By general results of [AT, Al] we have

THEOREM 1.1 (Alexandrino and Töben). *Any non-trivial polar action on a simply connected manifold has no exceptional orbits and its reflection group is the whole polar group.*

We assume from now on that M is simply connected.

It is clear that W acts transitively on the set of open chambers of Σ , but the stabilizer group W_c which we will call the *chamber group* may be non trivial when the section is not simply connected (cf. 1.2 and 1.3). However, since there are no exceptional orbits it follows that it must act freely on c .

Clearly

- $\Sigma/W = C/W_c$

Moreover, the boundary $\partial C = C - c$ of a chamber C , is the union of its *chamber faces*, where a chamber face is a non-empty intersection $C \cap \Lambda$ with a mirror. Clearly,

- $\partial(\Sigma^*) = \partial C/W_c$

and each face of $\partial\Sigma^*$ is the image of a (component of a) chamber face, but in general more than one chamber face gets identified with a face of $\partial\Sigma^*$ via the action of W_c . It is clear that the intersection of all chamber faces is contained in the fixed point set of W , but in general this fixed point set is larger. We also point out that although $\Sigma^* = C/W_c$ is an Alexandrov space (even an orbifold), C is not in general.

The following examples illustrate these concepts and are relevant for our subsequent discussion about positive curvature.

Example 1.2. Consider the following groups W acting on \mathbb{S}^2 as well as on \mathbb{RP}^2 .

(1) $W = A_1 = \langle r \rangle$, where r is the reflection in the equator:

On \mathbb{S}^2 there is one mirror and two open chambers, the open upper and lower hemispheres interchanged by r . Their closure is the orbit space \mathbb{S}^2/W . There is one face, its boundary circle (and it coincides with the mirror of r).

On \mathbb{RP}^2 there is one mirror and one open chamber and it is preserved by r . Its closure is all of \mathbb{RP}^2 and the orbit space \mathbb{RP}^2/W is the cone on its boundary circle, the cone point corresponds to the isolated fixed point of r on \mathbb{RP}^2 (in the chamber). There is one face, the whole boundary (and it coincides with the mirror of r).

Note that the action of W on \mathbb{RP}^2 lifts to the action of W on \mathbb{S}^2 . If we extend this action by $-\text{id}$, the extended group action induces the same action on the base, and now has the same orbit space.

(2) $W = A_1 \times A_1 = \langle r_0, r_2 \rangle$ where r_0, r_2 are reflection in two great circles making an angle $\pi/2$:

On \mathbb{S}^2 there are two mirrors and four open chambers. Their closure is the orbit space \mathbb{S}^2/W , a spherical right angled biangle. There are two faces each of which are also a chamber face. Their intersection is the intersection of mirrors and coincides with the fixed point set $\text{Fix}(W)$.

On \mathbb{RP}^2 there are three mirrors and four open chambers. In fact, the “rotation” $r_0 r_2$ on \mathbb{S}^2 induces a reflection on \mathbb{RP}^2 . The closure of an open chamber is the orbit space \mathbb{RP}^2/W , a right angled spherical triangle. There are three faces, each of which is also a chamber face. The intersection of all mirrors is empty, but each vertex of the orbit space triangle correspond in this case to a fixed point of W .

In this case, the lifted action of W on \mathbb{RP}^2 to \mathbb{S}^2 contains a rotation of angle π . Again the extended action by $-\text{id}$ defines the same action on \mathbb{RP}^2 , but on \mathbb{S}^2 the action has three reflections, and of course the same orbit space. In other words, the reflection group on \mathbb{S}^2 generated by the lift of all the reflections in \mathbb{RP}^2 contains the antipodal map in this case as opposed to the first case.

(3) $W = A_2 = \langle r_0, r_3 \rangle$ where r_0, r_3 are reflection in two great circles making an angle $\pi/3$:

On \mathbb{S}^2 there are three mirrors and six open chambers. Their closure is the orbit space \mathbb{S}^2/W , a spherical biangle with angle $\pi/3$. There are two faces each of which are also a chamber face. Their intersection is the intersection of mirrors and coincides with the fixed point set $\text{Fix}(W)$.

On \mathbb{RP}^2 there are three mirrors and three open chambers. The closure of an open chamber is a spherical biangle with angle $\pi/3$ where the two vertices have been identified! The stabilizer W_c of a chamber has order two, fixes the “mid point” of C and rotates C to itself, mapping one chamber face to the other. The orbit space has one face with one singular point, the fixed point of W and one interior singular point, the fixed point of W_c .

In this case, the reflection group obtained by lifting the reflections in \mathbb{RP}^2 to reflections in \mathbb{S}^2 does not contain the antipodal map. If we extend it by the antipodal map we get the same action on \mathbb{RP}^2 and the orbit spaces are of course the same as well.

Note that if the chamber group W_c is trivial, it follows that $C = \Sigma/W$, and that W acts simply transitively on the set of closed chambers of a fixed section Σ . Moreover, G acts transitively on the set of all chambers in all sections of M , i.e. $M = \bigcup_{g \in G} C$, and this set of chambers is G/H as a set. The chamber faces $F_i, i = 1, \dots, k$, of C correspond to a set of generators r_i for W . This way all faces of all chambers of M in all sections gets labelled consistently, so that G is label preserving. Now define two chambers (whether in a fixed section or not) to be *i-adjacent* if they have a common i face. This relation among the set of chambers in a fixed section Σ or among all chambers in M make both of these sets into a *chamber system* $\mathcal{C}(\Sigma, W)$, respectively $\mathcal{C}(M, G)$ according to the following definition (see, e.g., [Ti2, Ro]):

An (*abstract*) *chamber system* over $I = \{1, \dots, k\}$ is a set \mathcal{C} together with a partition of \mathcal{C} for every $i \in I$. Points $C, C' \in \mathcal{C}$ in the same part of the i -partition, are said to be *i-adjacent* which is written as $C \sim_i C'$. The elements of \mathcal{C} are called *chambers*.

We will use the following standard terminology in subsequent sections:

A *gallery* in \mathcal{C} is a sequence $\Gamma = (C_0, \dots, C_m)$ in \mathcal{C} such that C_j is i_j -adjacent to C_{j+1} for every $0 \leq j \leq m-1$. Here the word $f = i_0 i_1 \dots i_{m-1}$ in I is referred to as the *type* of the gallery. If we want to indicate this type, we write Γ_f rather than just Γ . If the i_j belong to a subset J of I , we call $\Gamma = (C_0, \dots, C_m)$ a *J-gallery*. A subset \mathcal{B} of a chamber system \mathcal{C} is said to be *connected* (or *J-connected*) if any two chambers in it can be connected by a gallery (or *J-gallery*). The *J-connected* components of \mathcal{C} are called *J-residues*. The *rank* of a *J-residue* is the cardinality of J ; its *corank* the cardinality of $I \setminus J$. Given residues R and S of types J and K respectively, we say that S is a *face* of R , if $R \subset S$ and $J \subset K$.

Note that for chamber systems $\mathcal{C}(M, G)$ as above, if two mirrors Λ_i and Λ_j in Σ corresponding to two reflections r_i and r_j on Σ intersect, then $(r_i r_j)^{m_{ij}} = 1$ for some finite integer $m_{ij} > 1$. In fact, $r_i, r_j \in W_p$ the reflection group of the polar representation of the isotropy group G_p for $p \in \Lambda_i \cap \Lambda_j =: \Lambda_{ij}$, so $\langle r_i, r_j \rangle$ is a dihedral group, and the angle between Λ_i and Λ_j is π/m_{ij} . In fact m_{ij} is limited to 2, 3, 4, or 6 when M is simply connected.

Recall, that a symmetric $k \times k$ matrix $M = (m_{ij})$ with entries from $\mathbb{N} \cup \{\infty\}$, with $m_{ii} = 1$ for all $i \in I$, and $m_{ij} > 1$ if $i \neq j$ is called a *Coxeter matrix*.

Pictorially, M is given by its so-called *diagram*, which consists of one *node* for each $i \in I$ and $m_{ij} - 2$ lines joining the i and j nodes.

The associated *Coxeter group of type M* is the group $W(M)$ given by generators and relations as

$$W(M) = \langle \{r_1, \dots, r_k\} \mid (r_i r_j)^{m_{ij}} = 1 \text{ for all } i, j \in I \text{ such that } m_{ij} \text{ is finite} \rangle.$$

The pair $(W(M), S)$, where $S = \{r_1, \dots, r_k\}$, is called the *Coxeter system of type M*, and k is referred to as its *rank*. The elements of $W(M)$ that are conjugate to elements in S are called *reflections*.

There is a natural chamber system, $\mathcal{C}(W)$ associated with a Coxeter system $(W(M), S)$, where $S = \{r_1, \dots, r_k\}$ and $I = \{1, \dots, k\}$: One defines i -adjacency for $i \in I$ to be $w \sim_i wr_i$, i.e. each part in the i -partition of W consists of two elements. Notice that W is connected since S generates W . There is a partial order among residues defined by $S \leq R$ if S is a face of R . The residues T for which $S \leq T$ implies $S = T$ are called the *vertices* of $\mathcal{C}(W)$. Denote the set of vertices by \mathcal{V} . One associates to a residue S the subset $S' \subset \mathcal{V}$ defined by $S' = \{v \in \mathcal{V} \mid v \leq S\}$, and call S' an *i-simplex* if its cardinality is $i + 1$. The set simplices in \mathcal{V} is denoted by $\Delta(W)$.

The Coxeter complex, $\Delta(W)$ associated to a Coxeter system $(W(M), S)$ also provides an example of an (abstract) simplicial complex:

Recall that an (*abstract*) *simplicial complex* is a nonempty family \mathcal{S} of finite subsets (called *simplices*) of a set V so that $\{v\} \in \mathcal{S}$ for every $v \in V$ and every subset of a simplex in \mathcal{S} is a simplex in \mathcal{S} (called a *face*). (The simplices consisting of one element are called *vertices*.)

One has the following general facts (see e.g. [D1]):

THEOREM 1.3. *For any reflection group W on a simply connected Riemannian manifold Σ , (W, S) is a Coxeter system, where S are reflections in W corresponding to the faces of a chamber C , and W_c is trivial. If Σ is compact, W is finite and isomorphic to a Euclidean reflection group.*

Remark 1.4. The Coxeter groups that we will deal with in this paper will all be finite. By a theorem of Coxeter, Coxeter systems of rank $k + 1$ are in one to one correspondence with finite subgroups of $O(k + 1)$ that are generated by reflections in hyperplanes of \mathbb{R}^{k+1} and only fix the origin. Such groups have been classified. Let (W, S) be a Coxeter system of rank k acting as a reflection group on \mathbb{R}^{k+1} , and consider its restriction to \mathbb{S}^k . In this case mirrors are of course great spheres \mathbb{S}^{k-1} , and the Coxeter group W acts simply transitively on the set of chambers. Each chamber is a spherical k -simplex and the corresponding triangulation of \mathbb{S}^k is the *geometric realization* of the Coxeter complex $\Delta(W)$ associated to a Coxeter system (W, S) .

The geometry of this representation is also reflected in the Coxeter diagram of M . For example, this diagram is connected if and only if this action is irreducible. Each node corresponds to a codimension one face simplex, i , and π/m_{ij} is the angle between the corresponding i and j faces of the k -simplex \mathbb{S}^k/W . Removing a node, yields the diagram for the residue of the corresponding face. In the dual interpretations where the roles of faces and vertices are reversed, the removal of a node yields the diagram for the isotropy group of W at the corresponding vertex.

We note that the chambers for the Coxeter system (W, S) in theorem 1.3 when Σ is a compact k manifold combinatorially are the same as the spherical k simplices of its representation above. Geometrically, it follows in particular that all angles in a chamber of Σ are the same as the corresponding angles in the spherical simplex.

Although A_2 is an irreducible Coxeter group, we point out that all the linear 3-dimensional representations presented in 1.2 above are reducible. We conclude this section with important examples of irreducible Coxeter groups:

Example 1.5. Finite Coxeter groups that are isomorphic to finite and irreducible reflection groups acting on \mathbb{R}^3 will play a special role in some of our proofs. There are three such groups that in the classification of finite Coxeter (or reflection) groups are given the symbols A_3, C_3, H_3 .

The group A_3 is isomorphic to the symmetric group on four letters. It is the group of symmetries of a regular tetrahedron. Its order is 24. The 2-simplexes in the triangulation explained above have angles $\pi/2, \pi/3$, and $\pi/3$ at the vertices.

The group C_3 is the symmetry group of a regular cube (or dually of a regular octahedron). Its order is 48. The 2-simplices in the triangulation have angles $\pi/2, \pi/3$, and $\pi/4$ at the vertices.

The group H_3 is the symmetry group of the regular dodecahedron (or dually to the regular icosahedron). Its order is 120. The 2-simplices in the triangulation have angles $\pi/2, \pi/3$, and $\pi/5$ at the vertices. Note, that the occurrence of the angle $\pi/5$ *excludes* H_3 as a Coxeter group of a polar action.

2. SECTIONS AND COXETER GROUPS

We assume from now on that M is equipped with a polar metric of positive sectional curvature. For short we will simply say that M is a *positively curved polar G -manifold*. This will yield strong restrictions on all the basic items presented in Section 1. In particular, we will prove that sections are either spheres or real projective spaces, also when M is not simply connected.

When M is simply connected, we show that the chamber group is trivial. It will follow that the polar group is a Coxeter group when the section is a sphere, and a \mathbb{Z}_2 quotient of such a group when the section is a real projective space, in either case the action is linear.

The starting point is the following

LEMMA 2.1 (Singular Orbit). *Any positively curved polar G -manifolds has singular orbits.*

Proof. If all orbits have maximal dimension, the normal distribution is globally defined and integrable with leaves the sections of M . Since in particular the sectional curvature of M is nonnegative, it now follows from Theorem 1.3 in [Wa] that the orbits of G are totally geodesic, and that the metric on M locally is a product metric. This is a contradiction since the sectional curvature of M is actually positive. \square

Remark 2.2. For a nonnegatively curved polar G -manifold the same conclusion holds unless $M = \Sigma \times_{\Pi} G/H$ is locally metrically a product. If in addition M is simply connected, $M = \Sigma \times G/H$ with product metric.

From Section 1 we know in particular that the reflection group $W \subset \Pi$ is nontrivial and that $\partial M^* = \partial \Sigma^*$ is non empty. This already is sufficient to prove

PROPOSITION 2.3 (Section). *Let M be a positively curved polar manifold. Then any section Σ is diffeomorphic to either a sphere \mathbb{S}^k or a real projective space \mathbb{RP}^k . In particular, the polar group Π is finite.*

Proof. Let r be a reflection, with mirror Λ and $E \subset \Lambda$ a component. Since the curvature is positive the (local) distance function to E is strictly concave. In particular, the complement $\Sigma - D_\epsilon(E)$ of a small tubular neighborhood of E is a (locally) convex set with boundary $\partial D_\epsilon(E)$. This set either has one or two components corresponding to the boundary having one or two components. In either case, each component is a disc by the standard “soul argument”, and in fact $E = \Lambda$. The key fact here is that the distance function to the boundary is strictly concave and hence has a unique point at maximal distance called the *soul point*. Moreover the distance function to the soul point has no critical points. For the arguments and constructions below it is also important that the distance function is r invariant.

In the case, where $\Sigma - D_\epsilon(E)$ has two components, $\Lambda = E$ separates Σ into two manifolds V_+ and V_- each with Λ as a totally geodesic boundary. In this case the isometry r interchanges V_+ and V_- . Moreover, the diffeomorphism ϕ say from the upper hemisphere \mathbb{D}_+^k of \mathbb{S}^k to V_+ can be chosen so that the north pole of \mathbb{D}_+^k goes to the soul point of V_+ , and the image of the gradient lines to the north pole of \mathbb{D}_+^k are “radial” near E and the soul point. The map $\Phi : \mathbb{S}^k \rightarrow \Sigma$ defined by $\Phi = \phi$ on \mathbb{D}_+^k and $\Phi = r\phi\rho$ on \mathbb{D}_-^k is a diffeomorphism which is equivariant relative to the reflections ρ in the equator of \mathbb{S}^k and r on Σ .

In the case where $\Sigma - D_\epsilon(E)$ has one components, r fixes its soul point and acts freely elsewhere: In fact r clearly acts freely in $D_\epsilon(E) - E$, so by convexity r can only have isolated fixed points in $\Sigma - D_\epsilon(E)$. Moreover, if there was an isolated fixed point in addition to the soul point a minimal geodesic between them would be reflected to a closed geodesic which is impossible by convexity. In particular, $\Lambda = E = \mathbb{RP}^{k-1}$ and Σ has fundamental group \mathbb{Z}_2 . In the two fold universal cover $\tilde{\Sigma}$ of Σ , the lift $\tilde{\Lambda}$ splits $\tilde{\Sigma}$ into two convex components V_+ and V_- with common totally geodesic boundary $\tilde{\Lambda}$, as in the first part. The reflection r lifts to a reflection \tilde{r} interchanging V_+ and V_- , each being mapped isometrically by the projection map to $\Sigma - \Lambda$ (it also lifts to a “rotation” preserving V_\pm and acting as a on $\tilde{\Lambda}$). Choosing a diffeomorphism ϕ say from the upper hemisphere \mathbb{D}_+^k of \mathbb{S}^k to V_+ as before, the map $\tilde{\Phi} : \mathbb{S}^k \rightarrow \tilde{\Sigma}$ defined by $\tilde{\Phi} = \phi$ on \mathbb{D}_+^k and $\tilde{\Phi} = r\phi\rho$ on \mathbb{D}_-^k is a diffeomorphism which is equivariant relative to the reflection $\tilde{\rho}$ on \mathbb{S}^k and r on $\tilde{\Sigma}$, and in addition by construction equivariant relative to the antipodal map $-\text{id}$ on \mathbb{S}^k and the deck transformation a of $\tilde{\Sigma}$. We conclude that $\tilde{\Phi}$ induces a diffeomorphism $\Phi : \mathbb{RP}^k \rightarrow \Sigma$ which is equivariant relative to the reflections ρ on \mathbb{RP}^k induced from $\tilde{\rho}$ and r on Σ . \square

Remark 2.4. During the proof of the result above we note in particular that if $r \in W$ is a reflection of the section Σ with mirror Λ , then:

- If Σ is a sphere, $\text{Fix}(r) = \Lambda$ and Λ is a codimension one sphere.
- If Σ is a projective space, $\text{Fix}(r) = \Lambda \cup s$, where s is the soul point at maximal distance to Λ , and Λ is a real projective space of codimension one.

In particular, *mirrors are connected*, and if $\Pi = \langle r \rangle$, the result above gives a complete equivariant description of (Σ, Π) .

The proof above also allows us to derive further information about the reflection group W and the corresponding open chambers and orbit space Σ/W :

LEMMA 2.5 (Sphere Chamber). *Assume Σ is a k -dimensional sphere, and c is an open chamber. Then*

- *Intersections of mirrors are spheres, and the closure C is a convex set in Σ .*

- *There are at most $k + 1$ chamber faces, and the intersection of all of them is $\text{Fix}(W)$.*
- *If there are $k + 1$ chamber faces, then C is a k -simplex, and $\text{Fix}(W) = \emptyset$.*
- *If there are $\ell + 1 < k + 1$ chamber faces, then C is the join of $\text{Fix}(W)$ with an ℓ -simplex.*

Moreover, C is a fundamental domain for W and $\Sigma/W = C$.

Proof. If there is only one mirror Λ corresponding to one reflection r , $W = \langle r \rangle = \mathbb{Z}_2$ and C is a closed convex disc with boundary $\Lambda = \text{Fix}(W)$ and indeed a join of $\text{Fix}(W)$ with a 0-simplex, the soul point s of C as we have seen.

Now consider any two reflections, $r_i, i = 1, 2$ with corresponding mirrors Λ_i . If $p \in \Lambda_{12} := \Lambda_1 \cap \Lambda_2 = \text{Fix}(\langle r_1, r_2 \rangle)$, clearly $r_i \in W_p$ the reflection group of the polar representation of the isotropy group G_p . In particular, $\langle r_1, r_2 \rangle$ is a dihedral group, and the angle between Λ_1 and Λ_2 is π/k for some integer k . In particular, the intersection Λ_{12} is a codimension one totally geodesic submanifold of either mirror Λ_i , and hence again by convexity is a sphere (two points when the mirrors are 1-dimensional).

In general, consider ℓ mirrors $\Lambda_1, \dots, \Lambda_\ell$ such that the inclusions of iterated intersections $\Lambda_{12} \supset \Lambda_{123} \supset \dots \supset \Lambda_{123\dots\ell}$ are all strict. Then each intersection is a totally geodesic submanifolds of codimension one in the previous intersection, and hence $\Lambda_{123\dots\ell}$ is a $(k - \ell)$ sphere. Clearly, also $\Lambda_{123\dots\ell}$ is the set fixed by all reflections r_i with corresponding mirror Λ_i . This completes the proof of the first two “bullets”, since mirrors corresponding to ℓ different chamber faces satisfy the needed inclusion property.

Now suppose C has $\ell + 1$ chamber faces, F_0, \dots, F_ℓ . Since the angle between any two faces is at most $\pi/2$ it is easy to see that the distance function on C to one face, say F_0 is strictly concave, and hence has a unique point at maximal distance, its “soul point”, s_0 . Clearly, s_0 is in the intersection of the remaining chamber faces. Moreover, the distance function to s_0 on C has no (geometric) critical points, and one easily constructs a *gradient like* vector field which is radial near s_0 and gradient like also when restricted to the remaining faces intrinsically. In particular, C is the cone on F_0 which in turn is isotopic to a small metric ball in C of radius ϵ centered at s_0 . This also identifies F_0 with the boundary of this ϵ ball, which via the exponential map is identified with the closure of a chamber in the unit sphere at s_0 corresponding to the reflections r_1, \dots, r_ℓ . The proof of the remaining two bullets is now completed by induction on the number of chamber faces. \square

We point out that this proof is a special case of a general result about orbit spaces of positively curved manifolds due to Wilking [Wi2] (related more directly to Σ/W in our context however). We have included it here not only to make the exposition more self contained but also because it illuminates the particular structure we have here.

We now turn to the case where the section Σ is a projective space. In this case, we will analyze the situation in its universal cover $\tilde{\Sigma}$. Specifically, for each mirror Λ in Σ corresponding to a reflection r , we consider its lift $\tilde{\Lambda}$ to $\tilde{\Sigma}$. From the proof of the above proposition, r has two canonical lifts. One of them is a reflection \tilde{r} in $\tilde{\Lambda}$, the other has two isolated fixed points and restricts to \mathfrak{a} on $\tilde{\Lambda}$. Here we define \tilde{W} to be the reflection group on $\tilde{\Sigma}$ generated by all \tilde{r} , where we use all r from W . Note, that by construction, any lifted mirror is preserved by \mathfrak{a} , and that \mathfrak{a} commutes with any element from \tilde{W} . Combining this with the previous lemma one derives, whether M is simply connected or not, the following:

LEMMA 2.6 (Projective Chamber). *Assume Σ is a k -dimensional projective space and $\tilde{\Sigma}$ the universal cover with deck transformation \mathbf{a} . Then*

- *Intersections of lifted mirrors are spheres invariant under \mathbf{a} .*
- *The associated reflection group \tilde{W} of W may or may not contain \mathbf{a} , but in either case $W = \langle \tilde{W}, \mathbf{a} \rangle / \langle \mathbf{a} \rangle$.*
- *Open chambers c in Σ are isometric to open chambers \tilde{c} for \tilde{W} ,*
- *The closure \tilde{C} of an open chamber for \tilde{W} is a convex set in $\tilde{\Sigma}$ with boundary the union of chamber faces. Moreover, C is obtained from \tilde{C} by identifying \mathbf{a} orbits in the boundary.*
- *\tilde{C} has at most $k + 1$ chamber faces, and the intersection of them all is $\text{Fix}(\tilde{W})$.*
- *If \tilde{C} has $k + 1$ chamber faces it is a k -simplex and $\text{Fix}(\tilde{W}) = \emptyset$.*
- *If \tilde{C} has $\ell + 1 < k + 1$ chamber faces it is a join of $\text{Fix}(\tilde{W})$ and an ℓ simplex.*

Moreover, $\Sigma/W = \tilde{\Sigma}/\langle \tilde{W}, \mathbf{a} \rangle = (\tilde{\Sigma}/\langle \tilde{W} \rangle)/\langle \mathbf{a} \rangle = \tilde{C}/\langle \mathbf{a} \rangle$.

The following is pivotal for us:

PROPOSITION 2.7 (Chamber Group). *The chamber group W_c of a simply connected positively curved polar \mathbf{G} manifold M is trivial, and hence $M^* = \Sigma^* = C$. Moreover,*

- *If Σ is a sphere, C is a simplex and $\text{Fix}(W) = \emptyset$, or C is a join of $\text{Fix}(W)$ and a simplex.*
- *If Σ is a projective space, C is a simplex, $\mathbf{a} \in \tilde{W}$ and $\text{Fix}(W)$ is a subset of the vertices (possibly empty).*

In either case, W acts simply transitive on the set of chambers.

Proof. Consider an open chamber c and W_c acting on it. Note that whether or not Σ is a sphere or a projective space, c is the union of compact closed locally convex subsets C^ϵ (distance ϵ or more to $C - c$). By convexity it is clear that the soul point (the common soul point s for all C^ϵ) is fixed by W_c (one can also use the description of c from the lemmas above). Since there are no exceptional orbits when M is simply connected (cf. 1.1) this already is impossible unless W_c is trivial. From section one we then know that $M^* = \Sigma^*$ is the closure C of a chamber c . If Σ is a sphere, lemma 2.5 completes the proof.

Now suppose Σ is a projective space:

First note that \mathbf{a} acts freely on the set of open chambers for \tilde{W} . This follows from the simple fact that \mathbf{a} interchanges the two connected components of the complement of any lifted mirror, and commutes with \tilde{W} .

We now claim that $\text{Fix}(\tilde{W}) = \emptyset$ and hence \tilde{C} is a simplex. Indeed, if $\text{Fix}(\tilde{W})$ is nonempty then clearly $\mathbf{a} \notin \tilde{W}$. Moreover, the involution induced by \mathbf{a} on $\tilde{\Sigma}/\tilde{W} = \tilde{C}$ acts freely on $\text{Fix}(\tilde{W})$ and preserves the boundary of \tilde{C} . In particular, $\tilde{C}/\langle \mathbf{a} \rangle$ will have interior metric singular points contradicting that it is C by lemma 2.6.

To complete the proof we now claim that $\mathbf{a} \in \tilde{W}$, and in particular $C = \Sigma/W = \tilde{\Sigma}/\tilde{W} = \tilde{C}$. If not, then $|W| = |\tilde{W}|$ and \mathbf{a} induces a non-trivial involution on \tilde{C} with $C = \tilde{C}/\langle \mathbf{a} \rangle$. Such an involution will preserve the boundary of the simplex \tilde{C} taking faces to faces. As before this will produce an interior metric singular point of C unless the induced map by \mathbf{a} is a reflection of the simplex. This, however, cannot happen since the fixed point set of this involution would correspond to a chamber face of C and hence a reflection in W whose lift to $\tilde{\Sigma}$ had been omitted from \tilde{W} . \square

Remark 2.8. Note that it follows from this that if $M^G \neq \emptyset$ and Σ is a sphere then $M^G = \Sigma^W$, since Σ^W is the most singular stratum in the orbit space $\Sigma/W = M/G$. In the next section we will see that conversely, if $\Sigma^W \neq \emptyset$ and Σ is a sphere then $M^G \neq \emptyset$ as well and hence $M^G = \Sigma^W$ (cf. (3.4)).

Since the Coxeter groups W , or \tilde{W} corresponding to the section being a sphere or a projective space respectively admit linear representations we get the following

COROLLARY 2.9. *For a simply connected positively curved polar G manifold M , $M^* = \Sigma^*$ admits a metric of constant curvature with the same infinitesimal metric singularities.*

Proof. From Proposition 2.7 we know that Σ^* is the chamber C for a Coxeter group W (resp. \tilde{W}) acting on the k -sphere Σ (resp. $\tilde{\Sigma}$, when Σ is a projective space). Moreover, the same Coxeter group acts linearly on \mathbb{S}^k , with chambers having labels as C and with the same infinitesimal singularities determined by corresponding isotropy groups and actions.

Now fix a chamber C with $\ell + 1$ chamber faces, F_0, \dots, F_ℓ in Σ (assuming w.l.o.g. it is a sphere), and a corresponding chamber in C' in \mathbb{S}^k . As in the proof of Lemma 2.5 let s_0 be the point in C at maximal distance to the face F_0 . Now apply the isotropy group W_{s_0} of the Coxeter group at s_0 to C to obtain a W_{s_0} invariant convex subset $W_{s_0}(C)$ of Σ with s_0 in the interior, the point at maximal distance from the boundary $\partial(W_{s_0}(C)) = W_{s_0}(F_0)$ of $W_{s_0}(C)$. As in the proof of Lemma 2.5 it follows that there is a W_{s_0} invariant smooth vector field on an open neighborhood of $W_{s_0}(C)$ in Σ , which is radial near s_0 and gradient like on $\partial(W_{s_0}(C))$.

The same construction based on C' in \mathbb{S}^k yields a W_{s_0} invariant diffeomorphism of a neighborhood of $W_{s_0}(C)$ in Σ to a neighborhood of $W_{s_0}(C')$ in \mathbb{S}^k . After a suitable reparametrization one of the vector fields using transversality if needed, the restriction yields the desired diffeomorphism from C to C' . \square

We are now ready to establish the main result of this section.

THEOREM 2.10 (Coxeter Section). *Let M be a simply connected positively curved polar manifold. Then the action of the polar group W of a section Σ is differentiably equivalent to a linear action of W . In fact, Σ admits a W invariant metric of constant curvature.*

Proof. Choose a constant curvature metric on Σ^* as above. We now claim that this metric comes from a W invariant metric on Σ with constant curvature. To see this all we have to do is to lift the metric locally near any point of the orbit space to any point mapping to it by the orbit map. This however is clear. Since the lifted metrics obtained this way agree on overlaps we are done. \square

We remark that in the literature the notion *hyperpolar* is used for a polar manifold with flat sections. Following [GZ] we say that a polar manifold is a *polar space form* if its sections have constant curvature. According to sign of the curvature of the sections one then says that the polar space form has spherical, euclidean or hyperbolic type. Using this language a simple partition of unity argument now immediately yields the following in our case

COROLLARY 2.11 (Polar Space Form). *A simply connected polar G manifold M admits the structure of a polar spherical space form with the same sections.*

It should be noted that M with such a polar space for structure typically has curvatures of both signs. In general, a highly non-trivial result of [Me] asserts that any metric on a section of any polar G manifold invariant under the polar group extends to a G invariant metric on the ambient manifold with the same section.

The following is now natural

Definition 2.12. We say that a simply connected positively curved polar G manifold M is *reducible* if the Coxeter system W is reducible.

In particular it follows that W , or \tilde{W} is an irreducible Coxeter system group when (M, G) is irreducible, but Example 1.2 implies that the converse is false. Also an action with a nontrivial fixed point set is reducible. In the case of irreducible actions all the types $A_n, C_n, D_n, E_6, E_7, E_8$ and F_4 are of course possible when the section is a sphere, induced by polar representations, but we note that due to the Chamber Group Proposition above, not all of them are possible when the section is a projective space.

3. THE CHAMBER SYSTEM AND PRIMITIVITY

Based on the Chamber group proposition 2.7, recall from section 1 that there are two natural chamber systems $\mathcal{C}(\Sigma, W)$, respectively $\mathcal{C}(M, G)$ associated with any polar action of a connected compact Lie group G on a simply connected positively curved manifold M with section Σ and polar group W . Throughout the rest of the paper (M, G) is such a polar pair.

Our primary purpose in this section is to analyse $\mathcal{C}(M, G)$ further and thereby derive essential properties about such general actions. In particular, we will show that this is a connected chamber system (the crucial starting point for our subsequent investigation of irreducible actions), and use this to show that G is generated by the face isotropy groups of any fixed chamber $C \subset \Sigma$ (an essential ingredient in our investigation of reducible actions).

When the chambers are simplices, we observe that all proper residues of the chamber system can be described via slice representations of corresponding isotropy groups. This allows us to invoke a celebrated result of Tits [Ti2] implying that the so-called universal cover of our chamber system is a building in most cases.

From the description $\mathcal{C}(M, G) = \cup_{g \in G} gC$ of the chamber system we first note that all chambers are isometric when equipped with the induced length space metric from M . This induces a natural length space metric on each path connected component of $\mathcal{C}(M, G)$. A fundamental Theorem due to Wilking [Wi3] asserts in particular that the *dual foliation* associated to the orbits of an isometric groups action on a positively curved manifold has only one leaf. It is an immediate consequence of this result that

- $\mathcal{C}(M, G)$ has only one component.

There is an equivalent length metric on $\mathcal{C}(M, G)$ obtained by using a polar space form metric on M (cf. 2.11) in the construction above. We will refer to the corresponding topology as the *thin* topology on $\mathcal{C}(M, G)$. (Since M is the union of its chambers, we can also think of it as M being equipped with this metric and topology).

From now on, we will always use the thin length metric on $\mathcal{C}(M, G)$ induced from a constant curvature one metric on a section. In particular, note that then each chamber C is either a

(spherical) k - simplex Δ^k , or else the spherical join $\mathbb{S}^{k-\ell-1} * \Delta^\ell$ of the $(k - \ell - 1)$ -sphere and a spherical ℓ - simplex. In either case, the chambers in a fixed section Σ tiles the section, which is either \mathbb{RP}^k or \mathbb{S}^k . Moreover, by construction, \mathbf{G} preserves the *labeling* of all “vertices, edges, \dots , faces”, i.e., of all 0-, 1-, \dots , $(k - 1)$ -simplices, when $C = \Delta^k$ is a simplex. In the special case where the chamber is not a simplex, i.e., $C = \mathbb{S}^{k-\ell-1} * \Delta^\ell$, by a “vertex”, or “0-simplex” of the chamber C we mean a set of the type $\mathbb{S}^{k-\ell-1} * \{v\}$, where v is a vertex of the simplex Δ^ℓ , and similarly for “edges, \dots , faces”. We label the set $\mathbb{S}^{k-\ell-1} \subset C$ as the -1 -simplex of the chamber C . In either case we note that the intersection of any two chambers in M is either empty or else a common “subsimplex” in this sense, allowing in particular the intersection to be a “ -1 -simplex”.

From the fact that $\mathcal{C}(M, \mathbf{G})$ with the thin topology is connected, we get the essential property:

THEOREM 3.1 (Connectivity). *Assume M is a 1-connected positively curved polar \mathbf{G} manifold. Then the associated chamber system $\mathcal{C}(M; \mathbf{G})$ is connected, i.e., any two chambers are connected by a gallery.*

Proof. We will prove this by induction on $\dim M^* = k$ using that $\mathcal{C}(M; \mathbf{G})$ is path connected. For simplicity we first present the proof in the typical case where the chamber C is a simplex Δ^k . A simple modification yields the general statement.

Let C and C' be two chambers of $\mathcal{C}(M; \mathbf{G})$. Using [Wi3] join two interior points of C and C' by a piecewise smooth *horizontal curve*, i.e., at any point both one sided derivatives of the curve are perpendicular to the \mathbf{G} orbit at the point. In our case, it is clear that we can choose a horizontal curve $\gamma : [0, 1] \rightarrow M$, and $0 = t_0 < t_1 < t_2 \dots < t_{k+1} = 1$ such that $\gamma|_{(t_i, t_{i+1})}$ is a geodesic, or once broken geodesic in the interior of a chamber C_i relative to the thin metric on $\mathcal{C}(M; \mathbf{G})$, where $C_0 = C$, $C_k = C'$ and all C_i are different. Moreover, γ can be chosen so that each of the possibly non smooth points $\gamma(t_i)$, $i = 1, \dots, k$ are all vertices. The normalized two opposite one sided derivatives of $\gamma(t_i)$ at the vertices are interior points of two $(k - 1)$ chamber simplices for the chamber complex $\mathcal{C}(\mathbb{S}_{\gamma(t_i)}^\perp; \mathbf{G}_{\gamma(t_i)})$ of the slice representation of the isotropy group $\mathbf{G}_{\gamma(t_i)}$. By induction these simplices are joined by a gallery in $\mathcal{C}(\mathbb{S}_{\gamma(t_i)}^\perp; \mathbf{G}_{\gamma(t_i)})$. Filling in the corresponding gallery in $\mathcal{C}(M; \mathbf{G})$ at each $\gamma(t_i)$ now yields a gallery from C to C' .

To complete the proof we need to establish the induction anchor in cohomogeneity two. By the same reasoning as above, this follows from the claim that the chamber complex of a linear spherical cohomogeneity one action is connected. Since any horizontal curve provided by Wilkings theorem in this case is a piecewise horizontal geodesic up to parametrization, such a curve already constitutes the desired gallery.

The modification needed to cover the case where the chambers are joins with a non-empty sphere can be explained as follows: As in the simplex case one may choose a piecewise horizontal geodesic γ , so that each of the possibly non smooth points $\gamma(t_i)$, $i = 1, \dots, k$ are most singular, i.e., in this case -1 -simplex points. The remaining part of the proof follows the same path. \square

The Coxeter Section Theorem 2.10 and the Connectivity Theorem above are the two crucial properties derived using positive curvature. We note that there is no reason for the chamber system of a simply connected polar space form of spherical type to be connected. However:

The manifolds we actually classify in higher cohomogeneities in this paper are the

Chamber Connected Polar Spherical Space Form

i.e.

- Simply connected polar space forms (M, \mathbf{G}) of spherical type
- with
- Connected associated chamber system, $\mathcal{C}(M; \mathbf{G})$

In addition, this generality is important for the proof, because \mathbf{G} invariant polar submanifolds of a positively curved polar manifold are typically not positively curved (cf. Section 7, proof of Theorem 7.1)

The two assumptions above will be applied throughout the rest of the paper.

Using connectivity we derive the following simple but powerful tool:

THEOREM 3.2 (Primitivity). *\mathbf{G} is generated by the face isotropy groups of any fixed chamber.*

Proof. Fix a chamber C_0 and consider any other chamber $\mathbf{g}C_0$, $\mathbf{g} \in \mathbf{G}$. Using the above, let $\Gamma = (C_0, \dots, C_k)$ be a gallery, of type $i_1 i_2 \dots i_k$, where $C_k = \mathbf{g}C_0$. By definition, note that any C_n is obtained from C_{n-1} by applying an element g_{i_n} of the isotropy group for the common face i_n of C_n and C_{n-1} to C_{n-1} , i.e., $C_n = g_{i_n} C_{n-1}$. From this it follows that $C_k = \mathbf{g}C_0 = g_{i_k} g_{i_{k-1}} \dots g_{i_1} C_0$, and hence $\mathbf{g} = g_{i_k} g_{i_{k-1}} \dots g_{i_1}$.

Now each g_{i_n} is a conjugate of an element of the isotropy group corresponding to the face i_n by the previous element. So in other words $g_{i_k} = [g_{i_{k-1}} \dots g_{i_1}] h_{i_k} [g_{i_{k-1}} \dots g_{i_1}]^{-1}$, and hence $\mathbf{g} = [g_{i_{k-1}} \dots g_{i_1}] h_{i_k} [g_{i_{k-1}} \dots g_{i_1}]^{-1} g_{i_{k-1}} \dots g_{i_1} = [g_{i_{k-1}} \dots g_{i_1}] h_{i_k}$, where h_{i_k} is in the isotropy group with face i_k of C_0 .

Proceeding in this way we see that $\mathbf{g} = h_{i_1} h_{i_2} \dots h_{i_k}$, where also $h_{i_1} = g_{i_1}$ as claimed. \square

Remark 3.3. Note that this also immediately implies that \mathbf{G} is generated by any two vertex isotropy groups. The proof also shows that \mathbf{G} is actually generated by the connected components of the face isotropy groups.

The description of galleries used in the proof above is very useful. In fact, a gallery starting at C of type $i_1 i_2 \dots i_k$ is given by a word $h_{i_1} h_{i_2} \dots h_{i_k}$ in elements of the isotropy groups \mathbf{G}_{i_j} corresponding to the i_j -faces of C . Note that each \mathbf{G}_{i_j} acts transitively on the normal sphere to the corresponding orbit stratum, i.e., the i_j residue of C is in one to one correspondence with this normal sphere. For this reason we say that a gallery $\Gamma_f = (C_0, \dots, C_k)$ of type $f = i_1 i_2 \dots i_k$ is obtained from C_0 by *folding* it repeatedly along faces using the face isotropy groups $\mathbf{G}_{i_1}, \mathbf{G}_{i_2}, \dots, \mathbf{G}_{i_k}$.

We can also use the above chamber connectedness to prove the fixed point claim from the previous section:

PROPOSITION 3.4. *Suppose M is a 1-connected positively curved polar \mathbf{G} manifold with spherical section Σ and polar group \mathbf{W} . Then $M^{\mathbf{G}} = \Sigma^{\mathbf{W}}$, and in particular $\text{rk}(\mathbf{W}) = \dim \Sigma^* + 1 - \dim M^{\mathbf{G}}$.*

Proof. Since obviously $M^{\mathbf{G}} \subset \Sigma^{\mathbf{W}}$ and equality has been proved in the previous section, it remains to prove that $M^{\mathbf{G}} \neq \emptyset$ as long as $\Sigma^{\mathbf{W}}$ is non empty.

By assumption $M^* = \Sigma^* = C = \Sigma^W * \Delta^\ell = \mathbb{S}^{k-\ell-1} * \Delta^\ell$. Since all G orbits corresponding to $\Sigma^W = \mathbb{S}^{k-\ell-1}$ are of the same type and are perpendicular to the section Σ it suffices to see that Σ^W is preserved by G .

Pick any $g \in G$ and join the chamber gC to C with a gallery. Since any two consecutive chambers in a gallery have a common “face” and thereby the same “ -1 -simplex”, i.e., the same fixed point set for the respective Weyl groups, it follows that also gC has the same “ -1 -simplex”, which however is $g\Sigma^W$. \square

Example 3.5. Here are examples showing that the conclusion above may fail in cohomogeneity one as well as when the section is a projective space.

(1) Let $M = \mathbb{CP}^n = \mathrm{SU}(n+1)/\mathrm{U}(n)$. Then $G = \mathrm{U}(n)$ acts by cohomogeneity one with one fixed point. However, its polar group is \mathbb{Z}_2 acting on a section \mathbb{S}^1 with two fixed points.

(2) The obvious polar $G = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(n)$ representation on $\mathbb{C}^{n+2} = \mathbb{C} + \mathbb{C} + \mathbb{C}^n$ descends to a polar action on \mathbb{CP}^{n+1} with two fixed points (corresponding to each \mathbb{C} summand). Its section is \mathbb{RP}^2 with $\mathbb{RP}^2/W = \mathbb{CP}^{n+1}/G$ a right angled spherical triangle. In particular, its Weyl group must necessarily have three fixed points.

The case where the orbit space $M^* = \Sigma^* = C$ is a join of a sphere and a simplex, and hence in particular $M^G \neq \emptyset$ will be dealt with in Section 7.

We now point out some simple but crucial strong local properties of the chamber system $\mathcal{C}(M, G)$ of a positively curved simply connected polar manifold in all remaining cases, i.e., when the orbit space is a simplex:

Say $M = (m_{ij})$, $i, j \in I$ is the Coxeter matrix for the reflection group W of the section Σ if it is a sphere, or else of \tilde{W} . In the latter case any word in the generators r_i of W whose lift is the antipodal map in \tilde{W} is a non-Coxeter relation in W , and must necessarily involve all generators of W . In particular:

For any fixed proper subset $J \subset I$ we have:

LEMMA 3.6. *The Coxeter group W_J is a subgroup of W (as well as of \tilde{W})*

Moreover, for any chamber C , consider $C_J := \cap_{i \in J} C_i$, where C_i is the i -face of C . For $p \in C_J$, let $\mathbb{S}_{p,J}^\perp$ denote the unit sphere normal to the orbit stratum of Gp at p . It is now apparent (see, e.g., 3.3) that

LEMMA 3.7. *The J -residue of \mathcal{C} and $\mathcal{C}(\mathbb{S}_{p,J}^\perp, G_p)$, for any $p \in C_J$ are equivalent as chamber systems of type M_J .*

Remark 3.8 (Polar Representations). The chamber system, $\mathcal{B} = \mathcal{C}(\mathbb{S}, K)$ associated to the restriction of a polar representation of a compact Lie group K to the unit sphere \mathbb{S} (without fixed points) is a fundamental example of a (spherical) Tits building.

Here a chamber system \mathcal{B} over I is called a *building* of type $M = (m_{ij})$, $i, j \in I$, if each chamber is i -adjacent to at least one other chamber, and there is a $W(M)$ valued “distance function”

$$\delta : \mathcal{B} \times \mathcal{B} \rightarrow W$$

with the property $\delta(x, y) = w \in W$ if and only if the types of minimal galleries between x and y coincide with the types of minimal galleries in the Coxeter complex $\mathcal{C}(\Sigma, W) =: \mathcal{W}$ from 1 to w .

The Coxeter complex \mathcal{W} being itself a building with $\delta(u, v) = u^{-1}v$, “isometric” images of \mathcal{W} in \mathcal{B} are called *apartments* in \mathcal{B} .

Remark 3.9 (Basic Building Properties). In a building \mathcal{B} , the following properties are basic and used repeatedly in the next sections:

- (Connectedness) Any two chambers x, y are joined by a minimal gallery Γ_f , which in turn is contained in an apartment A .
- (Uniqueness) A minimal gallery from x to y is uniquely determined by its type.
- (Convexity) If x, y are chambers in an apartment A , every minimal gallery from x to y is contained in A .
- (Homotopy) If Γ is a gallery from x to y of type f (not necessarily minimal), and $f \simeq g$ (see below), then there is a gallery of type g from x to y .
- A gallery of type f is minimal if and only if $f = i_1 \cdots i_m$ is a so-called *reduced word*, or equivalently $w = r_f := r_{i_1} \cdots r_{i_m}$ cannot be expressed as r_g for g a shorter word.

Since the slice representation of each isotropy group G_p is polar, it follows from 3.8 and the lemmas above that

PROPOSITION 3.10. *For any proper $J \subset I$, any J residue in the chamber system $\mathcal{C}(M, \mathbf{G})$ is a spherical building of type M_J .*

By invoking the following corollary of a profound result of Tits [Ti2] (cf. also [Ro])

THEOREM 3.11 (Tits). *The universal cover $\tilde{\mathcal{C}}$ of a (gallery-) connected chamber system \mathcal{C} of (finite) type M over I is a building if and only if all residues of cardinality three are covered by buildings.*

We conclude

THEOREM 3.12 (Building Cover). *Suppose M is a positively curved polar \mathbf{G} manifold with associated Coxeter system W of rank $\dim M^* + 1 \geq 3$. Then the universal cover $\tilde{\mathcal{C}}(M; \mathbf{G})$ of the associated chamber system $\mathcal{C}(M; \mathbf{G})$ is a spherical building except possibly when W has type C_3 .*

The fact that all residues of the rank at least 3 chamber system $\mathcal{C}(M; \mathbf{G})$ are buildings implies that its so-called universal cover $\tilde{\mathcal{C}}(M; \mathbf{G})$ can be viewed in two different equivalent ways: On the one hand, one can use the thin topology on $\mathcal{C}(M; \mathbf{G})$ and consider its usual topological universal cover. This cover clearly inherits the structure of a chamber system, and its fundamental group acts freely as a group of automorphisms on it. Alternatively, one can use the concept of *homotopies* among galleries developed by Tits, and use this in analogy with the usual construction of a universal cover.

Here two galleries $\Gamma_1\Gamma_0\Gamma_2$ and $\Gamma_1\Gamma'_0\Gamma_2$ in a chamber system, \mathcal{C} of type M over I are said to be *elementary homotopic* if Γ_0 and Γ'_0 are galleries in a rank 2 residue with the same extremities. A *homotopy* from a gallery Γ to another one Γ' (with fixed extremities) is a finite sequence of elementary homotopies which transforms Γ to Γ' . When such a homotopy exists we write $\Gamma \simeq \Gamma'$.

By construction, $\tilde{\mathcal{C}}$ as a set is a union of chambers, each chamber, $\tilde{C} \in \tilde{\mathcal{C}}$ being a *homotopy class*, $[\Gamma] = [C_0, \dots, C_m]$ of galleries $\Gamma = (C_0, \dots, C_m)$ from \mathcal{C} starting at a fixed chamber $C_0 \in \mathcal{C}$ and ending at $C_m = C \in \mathcal{C}$, and where the covering map $p : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ takes \tilde{C} to C_m . Also, the *adjacency* relation among chambers is defined as follows: $\tilde{C} = [C_0, \dots, C_m]$ is “*i*-adjacent” to $\tilde{C}' = [C_0, \dots, C_{m-1}, C'_m]$ when C_m and C'_m are “*i*-adjacent”, and to $\tilde{C}'' = [C_0, \dots, C_m, C'']$ for other *i*’s when C_m and C'' are “*i*-adjacent”. All other *incidence* relations follows from this, and the covering map p preserves incidence relations. The *i*-face of \tilde{C} is by definition the intersection of \tilde{C} with any \tilde{C}' that is *i*-adjacent to \tilde{C} . In this fashion the covering map p preserves faces, and hence all other types.

Note that in a Coxeter complex, \mathcal{W} galleries starting at 1 are in one-to-one correspondence with their types. Here one also uses the notion of *strict homotopy*, denoted $f \simeq g$, where elementary homotopies as above are replaced by the stronger notion of *strict elementary homotopy*. Here a strict elementary homotopy is an alteration from a word of the form $f_1 p(i, j) f_2$ to the word $f_1 p(j, i) f_2$, where $p(i, j)$ is a word of the form $\dots i j i j$ (with m_{ij} letters ending in *j*); e.g., if $m_{ij} = 3$, $p(i, j) = j i j$; and $p(j, i) = i j i$. In particular, f and g have the same length if they are strictly homotopic (but not if they are just homotopic). Also $r_f = r_g$ if f and g are strictly homotopic, but the converse is false, since one may have redundant letters, e.g, the words $f = f_1 i i f_2$ and $g = f_1 f_2$ are not strictly homotopic but $r_f = r_g$. A word f is called *reduced* if it is not strictly homotopic to a word of the form $f_1 i i f_2$.

Remark 3.13. We point out that buildings are simplicial complexes, but our chambers systems $\mathcal{C}(M; \mathbf{G})$ are frequently not. This is illustrated for example with the T^2 action on \mathbb{CP}^2 . Here all chambers are spherical right angled 2-simplices, and they all have the same three vertices, the fixed points of T^2 . Although we do not need it here, we can prove that in all irreducible cases, $\mathcal{C}(M; \mathbf{G})$ is indeed simplicial, only assuming that the sections are as in this paper.

Remark 3.14. We point out that equipped with the thin metric, our chamber system $\mathcal{C}(M, \mathbf{G})$ has the local structure of a CAT(1) space. This is of course true for its universal cover $\tilde{\mathcal{C}}(M, \mathbf{G})$ as well. In fact, when its dimension is at least three (corresponding to rank at least four), it follows by work of Charney and Lytchak [CL], that in fact $\tilde{\mathcal{C}}(M, \mathbf{G})$ is a CAT(1) space and in fact a spherical building by their geometric characterization of buildings.

4. COMPACT SPHERICAL BUILDINGS

Throughout this section, we assume that the orbit space $M^* = C$ is a simplex, and that the universal cover $\tilde{\mathcal{C}} := \tilde{\mathcal{C}}(M, \mathbf{G})$ of our base chamber system $\mathcal{C} := \mathcal{C}(M, \mathbf{G})$ is a spherical building of rank at least 3. We use $p : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ to denote the covering map.

Our primary objective is to endow $\tilde{\mathcal{C}}$ with a natural topology inherited from the topology of M , in such a way that it becomes a compact spherical building in the sense of Burns and

Spatzier [BSp] as extended by Grundhöfer, Kramer, Van Maldeghem, and Weiss in [GKMW]. Our second objective is to analyze the fundamental group π of $\mathcal{C}(M, \mathbf{G})$ and its action on the cover when $\mathcal{C}(M, \mathbf{G})$ is a compact spherical building. This in fact will imply Theorem A in the introduction in all cases except where \mathbf{G} has fixed points or where the Coxeter diagram for M either has isolated nodes or is of type \mathbf{C}_3 .

Sections 5 and 6 are devoted to the case where the Coxeter diagram of M is of type \mathbf{C}_3 . In the special reducible cases where isolated nodes are present in the Coxeter diagram of M or $M^{\mathbf{G}} \neq \emptyset$, rather different arguments will be employed in Sections 7 and 8.

We will write the set of vertices $\text{Vert}(\tilde{\mathcal{C}})$ of a Tits building $\tilde{\mathcal{C}}$ as a disjoint union $\text{Vert}(\tilde{\mathcal{C}}) = V_1 \cup \cdots \cup V_{k+1}$ over the vertices of the same cotype where $k+1$ is the rank of M . The set of r -simplices of type (i_1, \dots, i_{r+1}) for $r \leq k$ will be denoted by $\tilde{\mathcal{C}}_{i_1, \dots, i_{r+1}}$.

Recall, that a *compact (spherical) building* according to [BSp] is a Tits building $\tilde{\mathcal{C}}$ with a Hausdorff topology on the set $\text{Vert}(\tilde{\mathcal{C}}) = V_1 \cup \cdots \cup V_{k+1}$ of all vertices such that the set $\tilde{\mathcal{C}}_{i_1, \dots, i_{r+1}}$ of all simplices of type (i_1, \dots, i_{r+1}) is closed in the product $V_{i_1} \cup \cdots \cup V_{i_{r+1}}$. With the induced topology on the k simplices $\tilde{\mathcal{C}}_{1, \dots, k+1}$, $\tilde{\mathcal{C}}$ is called *compact, locally connected, infinite, metric* if $\tilde{\mathcal{C}}_{1, \dots, k+1}$ has the appropriate property.

It is the main result of [BSp] that an infinite, irreducible, locally connected, compact, metric, topologically Moufang building of rank at least 2 is classical. Namely, it is a Tits building associated to a non-compact real semisimple Lie group via the following description:

Example 4.1 (Symmetric Spaces and Buildings). Let U be a connected non-compact real semisimple Lie group without center and $K \subset U$ a maximal compact subgroup (which is unique up to conjugation). The isometric action of U on the *symmetric space* $N = U/K$ of non-positive curvature induces a continuous action on the boundary at infinity, \mathbb{S}_{∞} , with the same orbits as those of the subaction by K . Here the action by K is topologically equivalent to the isotropy representation of K on the unit sphere \mathbb{S}_p at $p \in N$ with $U_p = K$.

The isotropy representation of $K = U_p$ is polar with sections the tangent spaces of *flats* through $p \in N = U/K$. These flats at infinity are *apartments* of a (topological) building, $\mathcal{C}(U)$ equivalent to $\mathcal{C}(\mathbb{S}_p, U_p)$. One gets all apartments in the building in this fashion by letting p go through all points of N . The group U is the identity component of the (topological) automorphism group $\text{Aut}_{\text{top}}(\mathcal{C}(U))$ of the building.

An algebraic description of $\mathcal{C}(U)$ can be given via the set of all parabolic subgroups of U : If A is a maximal \mathbb{R} -split torus of U , the set Σ_A consisting of all parabolic subgroups of U containing A is an apartment of the building. The collection of all apartments Σ_A defines an apartment system, denoted by \mathcal{A} . If $C_1, C_2 \in \mathcal{C}(U)$, we call C_1 a *face* of C_2 and write $C_1 \leq C_2$ if $C_2 \supset C_1$. This partial order on $\mathcal{C}(U)$, together with \mathcal{A} , makes $\mathcal{C}(U)$ into a Tits building.

The correspondence between the geometric and algebraic description is that the isotropy groups under U of the chambers and their subsimplices at infinity are exactly the parabolic subgroups of U .

The topology on $\mathcal{C}(M, \mathbf{G})$

When considering $\mathcal{C}(M, \mathbf{G})$ as a set of chambers, each being a compact subset of the metric space M , $\mathcal{C}(M, \mathbf{G})$ is a compact metric space with the classical Hausdorff metric. Moreover, the same holds for the set of all galleries with any upper bound on the number of chambers. Since

$\tilde{\mathcal{C}}(M, \mathbf{G})$ is a building of type M any two chambers can be connected by a gallery of length at most $1/2|W(M)|$.

Let us fix a chamber $\tilde{C}_0 \in \tilde{\mathcal{C}}$. For any fixed large positive integer $k \geq \frac{1}{2}|W(M)|$, $\epsilon > 0$ and any chamber $\tilde{C} \in \tilde{\mathcal{C}}$, we let:

$$B_{\epsilon,k}(\tilde{C}) \text{ be the union of those chambers } \tilde{C}' \in \tilde{\mathcal{C}}$$

for which there are (stuttering) galleries Γ and Γ' of length at most k starting at \tilde{C}_0 and ending at \tilde{C} , respectively \tilde{C}' so that the (stuttering) galleries $p(\Gamma)$ and $p(\Gamma')$ in \mathcal{C} are within Hausdorff distance ϵ from one another in M . We will refer to the topology generated by these sets as the *chamber topology* on the building $\tilde{\mathcal{C}}$.

The geometric realization of the building $\tilde{\mathcal{C}}(M, \mathbf{G})$ is a simplicial complex $\tilde{\mathcal{S}}(M, \mathbf{G})$. We will show that this topology induces a topology on $\tilde{\mathcal{S}}(M, \mathbf{G})$ making it into a compact spherical building in the sense of Burns and Spatzier.

The following will be used repeatedly

LEMMA 4.2 (Homotopy Control). *Let Δ be a building of rank at least 3. Then for any k there is a $C(k)$ with the following property: Any galleries Γ and Γ' of lengths at most k with the same extremities are homotopic by a homotopy consisting of at most $C(k)$ chambers.*

Proof. Since any building of rank at least three is simply connected, Γ and Γ' are homotopic. The remaining part of our claim is proved by induction on k , being trivially true for $k = 1$.

If Γ and Γ' are both minimal, the claim is a direct consequence of the Convexity Property in 3.9. Similarly, if $\Gamma = \Gamma_1\Gamma_0\Gamma_2$ and $\Gamma' = \Gamma_1\Gamma'_0\Gamma_2$, where Γ_0 and Γ'_0 are minimal (e.g., when there is a strict elementary homotopy from Γ to Γ'). In particular, by induction it suffices to prove that a non-minimal Γ is strictly homotopic to a Γ' via an a priori bounded number of strict elementary homotopies and Γ' is homotopic to a shorter gallery within a uniformly bounded number of chambers.

Suppose Γ is not minimal of type f . We claim that Γ is strictly homotopic to a gallery Γ' of type $f_1 i f_2$ through at most ℓ^ℓ strictly elementary homotopies, where $\ell = |I|^k$. Indeed, the number of words of length at most k is bounded above by $|I|^k$. Therefore, the non-circuit operations from a word of length at most k to another one of length at most k is bounded above by ℓ^ℓ .

Now, a gallery Γ' of type $f_1 i f_2$ from x to y is obviously homotopic to a shorter gallery of type either $f_1 i f_2$ or type $f_1 f_2$, according to the chambers being $\Gamma_1 C_1 C_2 C_3 \Gamma_2$ (where $C_1 \sim_i C_2$, and $C_2 \sim_i C_3$) or $\Gamma_1 C_1 C_2 C_1 \Gamma_2$ (where $C_1 \sim_i C_2$). Moreover, the homotopy can be realized in the longer gallery and so the number of chambers is bounded by the length k . \square

Remark 4.3. The proof of the above lemma gives an algorithm to construct a *controlled homotopy* between galleries with the same extremities in a building.

PROPOSITION 4.4. *With the chamber topology, $\tilde{\mathcal{C}}$ is a compact, separable and metrizable space.*

Proof. By the Uryson Characterization Theorem for metrizable spaces, all we need to prove is that $\tilde{\mathcal{C}}$ is sequentially compact, separable, and regular.

- (Sequential Compactness) Any sequence $\{\tilde{C}_n\}$ of chambers in \mathcal{C} has a convergent subsequence.

For each n , let Γ_n be a galleries of length at most k joining \tilde{C}_0 and \tilde{C}_n . By compactness of M the sequence $p(\Gamma_n)$ has a convergent subsequence in the Hausdorff metric topology with limit a gallery $\bar{\Gamma}_\infty$ starting at $p(\tilde{C}_0)$. By the unique homotopy lifting property (cf. Ronan, Lemma 4.4), $\bar{\Gamma}_\infty$ can be uniquely lifted to a gallery, say Γ_∞ , starting at \tilde{C}_0 . By the definition of the chamber topology we know that the corresponding subsequence of $\{\tilde{C}_n\}$ converges to the end chamber of Γ_∞ .

- (Separability) We may choose a countable dense subset Q_i of each face isotropy group G_i , e.g. the rational points. The set of galleries starting at \tilde{C}_0 of length at most k obtained by the folding process described in 3.3 using only elements from Q_i is clearly dense in the set of all galleries starting at \tilde{C}_0 of length at most k . By definition, the last chamber of these lifted galleries in \mathcal{C} starting at \tilde{C}_0 form a countable dense set in the chamber topology.

- (Regularity) We need to prove that, for a chamber \tilde{C}_1 and a closed subset $B \subset \mathcal{C}$ in the complement of \tilde{C}_1 , there are two disjoint open sets U and V containing \tilde{C}_1 and B respectively.

If this is not the case, we find for arbitrary large integers n , a chamber $\tilde{C}'_n \in B_{\frac{1}{n},k}(\tilde{C}_1) \cap B_{\frac{1}{n},k}(B)$. By the above we know that the closed subset B is sequentially compact. Therefore, a subsequence of \tilde{C}'_n converges to some chamber $\tilde{C}_2 \in B$. Therefore, there are two pairs of sequences of galleries $\Gamma_{i,n}, \Gamma'_{i,n}$, $i = 1, 2$, starting at \tilde{C}_0 and ending at the chambers \tilde{C}_1, \tilde{C}_2 , respectively \tilde{C}'_n , with $d_H(p(\Gamma_{i,n}), p(\Gamma'_{i,n})) < \frac{1}{n}$. For each n , $\Gamma'_{1,n}$ and $\Gamma'_{2,n}$ have the same extremities in the building $\tilde{\mathcal{C}}$, and hence $\Gamma'_{1,n} \simeq \Gamma'_{2,n}$ and $p(\Gamma'_{1,n}) \simeq p(\Gamma'_{2,n})$, by a homotopy H'_n . By Lemma 4.2 we can assume that H'_n is composed of an a priori bounded number of chambers independent of n . Taking convergent subsequences, we can assume that $p(\Gamma_{i,n})$ as well as $p(\Gamma'_{i,n})$ converge to the same galleries $\bar{\Gamma}_{i,\infty}$, and that these are homotopic by a homotopy H'_∞ . So on the one hand, by the unique homotopy lifting property, $\bar{\Gamma}_{i,\infty}, i = 1, 2$ lift to galleries with the same end chamber in \mathcal{C} . On the other hand they lift to galleries with end chamber $\tilde{C}_i, i = 1, 2$ respectively. A contradiction. \square

LEMMA 4.5 (Independence). *The chamber topology is independent of the choices of \tilde{C}_0 and the parameter k .*

Proof. Let us first prove the independence of k . If $k' > k$ clearly $B_{\epsilon,k}(\tilde{C}) \subset B_{\epsilon,k'}(\tilde{C})$. Consequently it suffices to show that a k' -convergent sequence of chambers $\{\tilde{C}_n\}$ is also k -convergent. By assumption there are galleries Γ_n and Γ'_n in \mathcal{C} of length at most k' starting at \tilde{C}_0 and ending at \tilde{C}_n respectively \tilde{C} such that the projected galleries $p(\Gamma_n)$ and $p(\Gamma'_n)$ Hausdorff converge to a gallery $\bar{\Gamma}_\infty$ (possibly stuttering) in \mathcal{C} . Again using Lemma 4.2 we see that the gallery Γ_n is homotopic to a gallery Γ''_n of length at most k by a homotopy H_n with an a priori bounded number of chambers. Note that $p(\Gamma''_n)$ subsequentially converges to a gallery $\bar{\Gamma}'_\infty = p(\Gamma'_\infty)$, where Γ'_∞ is the subsequential limit of Γ'_n . We may assume the homotopies $p(H_n)$ also converge, and therefore we get a limit homotopy between the two limit galleries $\bar{\Gamma}_\infty$ and $\bar{\Gamma}'_\infty$. By the homotopy uniqueness lifting property once again we get that Γ_∞ and Γ'_∞ have the same ending chambers \tilde{C} . Therefore, $\{\tilde{C}_n\}$ also k -converges to \tilde{C} .

To see the independence of the choice of \tilde{C}_0 join another chamber \tilde{C}'_0 to \tilde{C}_0 with a fixed gallery Γ_0 , and the claim follows from independence of k via concatenation with Γ_0 and its opposite. \square

We will now investigate the topology induced on the set of vertices from the chamber topology. That topology in turn will induce a topology on the simplicial complex $\mathcal{S}(M, \mathbf{G})$ associated to the building referred to as the *thick* topology. Assuming our chamber system $\tilde{\mathcal{C}}$ has rank $k+1$ corresponding to cohomogeneity k , for any $i \in I = \{0, \dots, k\}$, consider the set \tilde{V}_i of cotype i vertices in $\tilde{\mathcal{C}}$. Let $\pi_i : \tilde{\mathcal{C}} \rightarrow \tilde{V}_i$ denote the obvious projection map. For each i , we equip \tilde{V}_i with the quotient topology.

LEMMA 4.6 (Vertex Space). *For any $i \in I$, the projection $\pi_i : \tilde{\mathcal{C}} \rightarrow \tilde{V}_i$ is an open map, and \tilde{V}_i is compact and Hausdorff. Moreover, for any $x \in \tilde{V}_i$, the fiber $\pi_i^{-1}(x) \subset \tilde{\mathcal{C}}$, is the residue $\text{Res}(x)$ in $\tilde{\mathcal{C}}$ which is compact, and the restriction of the covering map $p : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ to this residue is a homeomorphism to the residue $\text{Res}(p(x))$ in \mathcal{C} .*

Proof. We begin with a proof of the last claim. By construction of $\tilde{\mathcal{C}}$, p provides an isomorphism between the residues as sub-buildings. We need to show that the chamber topology restricted to the residue $\text{Res}(x)$ coincides with the Hausdorff topology of $\text{Res}(p(x))$ in the manifold M .

Since $\tilde{\mathcal{C}}$ and \mathcal{C} are both compact and Hausdorff, and $p : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ obviously is continuous, it remains to check that $\text{Res}(x)$ is closed in $\tilde{\mathcal{C}}$. Let $\{\tilde{C}_n\}, n = 1, 2, \dots$ be a sequence of chambers in $\text{Res}(x)$ which converges in $\tilde{\mathcal{C}}$. Join a fixed chamber \tilde{C}_0 to \tilde{C}_1 by a gallery Γ . Using that the residues are buildings, join each \tilde{C}_1 to \tilde{C}_n by a minimal gallery Γ_n within the residue. A subsequence of the projection to \mathcal{C} of the concatenated galleries clearly converges in the Hausdorff topology, and the end chamber of the lift of the limiting gallery is the limit of $\{\tilde{C}_n\}$, which as a consequence is in the residue.

To show that \tilde{V}_i is Hausdorff it suffices to show that $\pi_i : \tilde{\mathcal{C}} \rightarrow \tilde{V}_i$ is an open map and the cotype i -adjacency is a closed relation, i.e. the subset

$$\{(\tilde{C}, \tilde{C}') \in \tilde{\mathcal{C}} \times \tilde{\mathcal{C}} : \tilde{C} \text{ and } \tilde{C}' \text{ have common cotype } i \text{ vertices}\}$$

is closed in the product topology. To show the latter, let $(\tilde{C}_n, \tilde{C}'_n)$ be a sequence converging to (\tilde{C}, \tilde{C}') , where $\pi_i(\tilde{C}_n) = \pi_i(\tilde{C}'_n)$. In particular C_n and C'_n share an i -vertex, and (C_n, C'_n) converges to (C, C') in the Hausdorff topology of M . Join \tilde{C}_n to \tilde{C}'_n by a minimal gallery Γ_n^i in the i -residue and pick a subsequence if necessary so that the image galleries $p(\Gamma_n^i)$ in the residues in M converges. Obviously the limit gallery joins C to C' , and in particular they share an i vertex. It follows that \tilde{C} and \tilde{C}' share the type i vertex.

Let us prove that π_i is open. For this we need to see that $\pi_i^{-1}(\pi_i(U))$ is open in $\tilde{\mathcal{C}}$, where U is a finite intersection of $B_{\epsilon_i, k}(\tilde{C}_i)$'s. Pick a chamber $\tilde{D}' \in \pi_i^{-1}(\pi_i(U))$, i.e. $\pi_i(\tilde{D}') = \pi_i(\tilde{C}')$ for some $\tilde{C}' \in U$. We need to find a neighborhood U' of \tilde{D}' so that for any $\tilde{D}'' \in U'$ there is a $\tilde{C}'' \in U$ with $\pi_i(\tilde{D}'') = \pi_i(\tilde{C}'')$.

Let V be a finite intersection of $B_{\epsilon_j, k}(\tilde{C}_j)$'s so that $\tilde{C}' \in V \subset U$. Since \tilde{C}' and \tilde{D}' are cotype i adjacent, they are in the same cotype i residue (of some vertex), and they can be joined within this residue by a gallery Γ explicitly obtained by folding (see 3.3) \tilde{C}' repeatedly along faces using face isotropy groups (fixing the cotype i vertex) in M via $p : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$. To complete the proof the following observation suffices: Consider the chamber $C' = p(\tilde{C}')$ in \mathcal{C} . Any chamber

Hausdorff close to C' is C' for some $g \in G$ close to $1 \in G$, and $g\Gamma$ is thus close to Γ . Thus this process and its inverse takes a neighborhood of \tilde{C}' to a neighborhood \tilde{D}' and conversely, and the claim follows. \square

Now we are ready to prove the first of our main results in this section.

THEOREM 4.7 (compact spherical building). *The spherical building $\tilde{\mathcal{C}}(M, G)$ with the topology on the set of vertices induced by the thick topology on the chambers is a compact spherical building if its rank is at least 3.*

Proof. We have seen in Lemma 4.6 that the space $V_1 \cup \dots \cup V_{k+1}$ of vertices is Hausdorff. It is therefore left to show that the set $\tilde{\mathcal{C}}_{i_1, \dots, i_{r+1}}$ of all simplices of type (i_1, \dots, i_{r+1}) is closed in the product $\tilde{V}_{i_1} \cup \dots \cup \tilde{V}_{i_{r+1}}$. It follows from Proposition 4.4 and Lemma 4.6 that the product map $\prod \pi_{i_j} : \tilde{\mathcal{C}}_{i_1, \dots, i_{r+1}} \rightarrow \tilde{V}_{i_1} \cup \dots \cup \tilde{V}_{i_{r+1}}$, for any multi-index i_1, \dots, i_{r+1} is continuous and its image is a closed subset, which finishes the proof. \square

It is clear from what we have proved so far that the compact spherical building in Theorem 4.7 is an infinite compact metrizable building. We can now apply the main results of [BSp] or rather its generalization in [GKMW] to compact spherical buildings that need not be locally connected.

THEOREM 4.8 (Classical Building). *Assume the compact spherical building $\tilde{\mathcal{C}}(M, G)$ has rank at least 3 and its associated Coxeter diagram has no isolated nodes. Then it is the building at infinity of a product of irreducible symmetric spaces N of noncompact type of rank at least 2. The topological automorphism group $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$ of the building $\tilde{\mathcal{C}}(M, G)$ is a real noncompact semisimple Lie group with finitely many connected components and its identity component is isomorphic to the identity component of the isometry group of the symmetric space N .*

Proof. It follows from Theorem 1.2 in [GKMW] that $\tilde{\mathcal{C}}(M, G)$ is the building at infinity of a product of irreducible symmetric spaces of rank at least 2 and a locally finite Bruhat-Tits building of dimension at least two. The building at infinity of the Bruhat-Tits building is totally disconnected and can therefore be excluded since by Lemma 4.6 the vertex residues are locally connected compact spherical buildings. The claims about $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$ follow from [BSp]. \square

We now prove a theorem in which we do not need to assume that there are no isolated nodes in the Coxeter diagram.

THEOREM 4.9 (Compact Transformation Group). *Assume the spherical building $\tilde{\mathcal{C}}(M, G)$ has rank at least 3 and is equipped with the thick topology. Then the deck transformation group π with the compact open topology is a closed subgroup of the topological automorphism group $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$. Moreover, there is a closed subgroup \tilde{G} of $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$, such that $\pi \subset \tilde{G}$ is a normal subgroup with quotient $\tilde{G}/\pi = G$, whose action covers the G -action on \mathcal{C} .*

Proof. It is a simple consequence of the Independence Lemma 4.5 that every element of π is a homeomorphism with respect to the chamber and thick topologies. In particular, π is a subgroup of the topological automorphism group $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$. We now prove that π is a closed subgroup of $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$. Let f_n be a sequence in π that converges to f in $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$ in the compact open

topology. Hence $f_n(\tilde{C})$ converges to $f(\tilde{C})$ in the chamber topology for every chamber $\tilde{C} \in \tilde{\mathcal{C}}$. Notice that $p(f_n(\tilde{C})) = p(\tilde{C})$. Therefore, $p(f(\tilde{C})) = p(\tilde{C})$, and it follows that f is in π .

It is well-known that the G -action on \mathcal{C} lifts to a covering group \tilde{G} -action on $\tilde{\mathcal{C}}$, where \tilde{G} fits in an extension (see [Ro], Exercise 8 in Chapter 4)

$$1 \rightarrow \pi \rightarrow \tilde{G} \rightarrow G \rightarrow 1.$$

Once again, by the Independence Lemma 4.5, we see that \tilde{G} is a subgroup of $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$ and as above one can check that it is closed. \square

Combining these results we have the following main result about irreducible polar manifolds of positive curvature:

THEOREM 4.10. *Any polar action of a compact connected Lie group G on a simply connected positively curved manifold M whose associated complex $\tilde{\mathcal{C}}(M, G)$ is a compact spherical building of rank at least three and whose diagram M contains no isolated nodes is equivariantly diffeomorphic to a polar action on a compact rank one symmetric space.*

Proof. It follows from Theorem 4.8 that the simplicial complex $\tilde{\mathcal{S}}$ as a set with the sum of the thin and thick topologies is a sphere that we will denote by \mathbb{S} . The closed subgroup π of $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$ is a Lie group since $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$ is a Lie group by Theorem 4.8. It follows that π is a compact Lie group since the orbits of π are clearly compact and the action free.

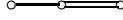
We would like to show that π is connected. We denote the identity component of π by π_0 . Clearly, π_0 acts freely on the sphere \mathbb{S} , and there is a covering $\mathbb{S}/\pi_0 \rightarrow \mathbb{S}/\pi = M$ whose fiber has the same number of points as π/π_0 . This is a contradiction since M is simply connected. It follows that $\pi = \pi_0$ and that π and \tilde{G} are both compact and connected subgroups of the identity component U of $\text{Aut}_{\text{top}}(\tilde{\mathcal{C}})$. As a consequence, \tilde{G} has a fixed point in the symmetric space U/K , where K is a maximal compact subgroup of the semisimple Lie group U . Therefore, up to conjugation we can assume that $\tilde{G} \subset K$ and it follows that the action by \tilde{G} is topologically equivalent to a linear polar action orbit equivalent to the isotropy representation of K on \mathbb{S} . The action of π on \mathbb{S} is linear and free. Hence π is either $\{1\}$, S^1 or S^3 . It follows that M is G -equivariantly homeomorphic to the rank one symmetric space \mathbb{S}/π with the linear polar action by $G = \tilde{G}/\pi$.

To complete the proof, we note that the induced linear polar action on \mathbb{S}/π by $G = \tilde{G}/\pi$ has the same data, i.e., section, polar group, isotropy groups and their slice representations as the polar G action on M . From the reconstruction theorem of [GZ] it follows that (M, G) is smoothly equivalent to $(\mathbb{S}/\pi, G)$. \square

5. HOMOGENEOUS C_3 -GEOMETRIES, A GENERAL APPROACH

Recall that a chamber system of type M is a chamber system over I for which each $\{i, j\}$ residue is a generalized m_{ij} -gon. In this section we consider homogeneous chamber systems \mathcal{C} of type C_3 with chamber transitive actions by a compact connected Lie group G . Let $|\mathcal{C}|$ be the geometric realization of \mathcal{C} . Let C denote a closed chamber in $|\mathcal{C}|$, which may be regarded as the G -orbit space. The three spherical angles of C are $\frac{\pi}{4}$, $\frac{\pi}{2}$, and $\frac{\pi}{3}$. Throughout the rest of this

and the next sections, we will use q , r , and t to denote the vertices at the $\frac{\pi}{4}$ -, $\frac{\pi}{2}$ -, and $\frac{\pi}{3}$ -angle respectively, which correspond to the three nodes from left to right of the C_3 -diagram



Let ℓ_q (resp. ℓ_r, ℓ_t) denote the opposite face of q (resp. r, t).

We can define an incident (Tits) geometry as follows: A vertex x is *incident* to a vertex y in $|\mathcal{C}|$ if x and y are contained in a closed chamber C . The *shadow* of a vertex x on the set of vertices of type i is the union of all vertices of type i incident to x . The incidence relation is preserved by the G -action. If $|\mathcal{C}|$ is a simplicial complex, then the Tits geometry is of type C_3 whose residues are isomorphic to the residues of \mathcal{C} .

Following Tits [Ti2], we call the vertices of type q , r and t *points*, *lines*, and *hyperlines* respectively. We denote by Q, R and T the set of points, lines, and hyperlines in \mathcal{C} . By the homogeneity of \mathcal{C} , the Lie group G acts transitively on Q, R and T .

By Tits [Ti2], Proposition 9, a connected C_3 Tits geometry is a building if and only if the following Axiom (LL) is fulfilled.

- (LL) If two lines are both incident to two different points, they coincide.

Note that there is a second axiom (O) in Proposition 9 of [Ti2] which trivially follows from (LL) if the chamber system is of type C_3 .

It is easy to see that

PROPOSITION 5.1. *If \mathcal{C} is a building of C_3 -type with a chamber transitive G -action, then the following holds:*

- ★ *for any pair of two different points $q, q' \in Q$ both incident to $x \in R$, we have*

$$G_q \cap G_{q'} \subset G_{xq} \cap G_{xq'}$$

Proof. Note that every line in the orbit of $G_q \cap G_{q'}$ at x is incident to both q and q' . Axiom (LL) implies that the orbit contains only one line. Hence $G_q \cap G_{q'} \subset G_x$. Since \mathcal{C} is a building, we have $G_x \cap G_q = G_{xq}$ and $G_x \cap G_{q'} = G_{xq'}$. The desired result follows. \square

From now on we will be concerned with the chamber system $\mathcal{C}(M, G)$ associated to a polar G -action of type M on a simply connected manifold M . But we remark that all arguments of this section work for Tits geometry of C_3 (resp. A_3) type whose associate field (from the residues of A_2 type) is $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or \mathbb{O} .

LEMMA 5.2. *Let M be a simply connected polar G -manifold whose associated chamber system $\mathcal{C}(M, G)$ is of type A_3 or C_3 . Then $|\mathcal{C}(M, G)|$ is a simplicial complex.*

Proof. A key step will be to prove

- For any two vertices of different types, there is at most one shortest geodesic connecting them.

This implies easily the desired result. Indeed, because of the claim, it suffices to show that there does not exist two chambers C_1, C_2 with all three faces the same. By the transitivity, $C_2 = gC_1$. Notice that g fixes all the three vertices and faces, hence g is in the principal isotropy group of the chamber C_1 , and so $C_2 = gC_1 = C_1$.

For simplicity we will only prove the above claim for C_3 type, and the A_3 type is easier and follows from the proof.

We first prove the claim when one of the vertices is a hyperline. For a hyperline $x \in T$, note that the shadow $\text{Sh}_Q(x)$ (resp. $\text{Sh}_R(x)$) of x in Q (resp. R) is a homogeneous space by G_x , say $G_x / G_x \cap G_q$. Moreover, the set of all faces containing x and q is the orbit $G_x \cap G_q / G_w$, where w is the tangent vector to a reference geodesic connecting x and q . It suffices to prove that $G_w = G_x \cap G_q$. If G_w is a proper subgroup, the fibration

$$\frac{G_x \cap G_q}{G_w} \rightarrow \frac{G_x}{G_w} \rightarrow \frac{G_x}{G_x \cap G_q}$$

has a fiber more than one point. Note that the base of the fibration can not be a point, since otherwise, $G_x \subset G_q$, and so by the primitivity $G = \langle G_x, G_q \rangle = G_q$, and hence G has fixed points. On the other hand, notice, G_x / G_w is the set of points (resp. hyperplanes) in a type A_2 geometry in the slice representation, hence $G_x / G_w = \mathbb{P}^2(k)$, where $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or \mathbb{O} . It is clear that the projective plane can not be a nontrivial fibration. The claim follows for $x \in T$.

It remains to prove the claim for the shadow $\text{Sh}_Q(x)$ where $x \in R$. As above, if w is the tangent vector to the reference shortest geodesic from x to q , then $G_x / G_w = \mathbb{S}^d$, where $d = 1, 2, 4, 8$ is the multiplicity of the opposite face of q . If $d \neq 1$, the even dimensional sphere can only be a nontrivial fibration with fiber \mathbb{Z}_2 . But if $d = 1$, the circle may be a fibration with any finite fiber.

If d is even, and the fibration has \mathbb{Z}_2 as the fiber. In other words, $G_w \subset G_x \cap G_q$ is an index 2 normal subgroup, there is an element $g \in G_x \cap G_q$ acting on the sphere \mathbb{S}^d by the antipodal map. Therefore, $g(w) = -w$. Hence, for the shortest geodesic γ from x to q , the union $\gamma \cup g\gamma$ is a geodesic which is smooth except possibly at q , and so $\gamma \cup g\gamma \subset \Sigma$, where Σ is a section. A contradiction.

Finally, we prove the claim for $d = 1$. Notice that the slice representation of G_x on T_x^\perp is reducible, i.e., $T_x^\perp = V_1 \oplus V_2$, where V_1 has dimension 2. Choose a chamber C in a section Σ such that the above geodesic $\gamma \subset C$. Let $t \in T$ denote the vertex of C other than x, q . Notice that $G_x = G_w G_{w'}$, where w' is the tangent vector to the shortest geodesic in Σ from x to t . If the above fibration has a fiber more than a point, i.e., $G_w \subset G_x \cap G_q$ is a proper subgroup of index greater than 1, there is an element $g \in G_x \cap G_q$ but $g \notin G_w$. Hence $g\gamma \neq \gamma$ for the geodesic as before. By multiplying an element of G_w if necessary, we may assume furthermore that $g \in G_{w'}$. Therefore, g fixes all the three vertices of C . Hence, $g \in G_x \cap G_q \cap G_t$, which equals to the principal isotropy group of the chamber, since $G_t \cap G_q = G_{tq}$ by the previous case, since there is no double connection between t and q . This implies that $gC = C$. A contradiction. \square

By Tits any A_n -geometry is a building. We assume from now on that $\mathcal{C}(M, G)$ is a geometry of C_3 type. For a chamber C , recall that the multiplicity triple, i.e., the dimension of the unit spheres in the normal slice, along the opposite face of q, r, t is $(d, d, k) \in \mathbb{Z}_+^3$, where $d = 1, 2, 4$ or 8 .

Let $x \in R$ be a line, and let $\mathbb{S}_{x,Q}^\perp$ be the normal sphere in the summand in the slice T_x^\perp whose image under \exp contains Q . Notice that, the shadow of a line $x \in R$ in Q is the image of $\mathbb{S}_{x,Q}^\perp$ under the normal exponential map. Moreover, the isotropy group G_x acts transitively on $\mathbb{S}_{x,Q}^\perp$. Let K_x denote the identity component of the kernel of the transitive G_x action on $\mathbb{S}_{x,Q}^\perp$. It is clear that the fixed point set M^{K_x} is a subcomplex that inherits an incidence structure.

LEMMA 5.3. *Let $\mathcal{C}(M, \mathbf{G})$ be a connected \mathbf{G} -homogeneous chamber system of \mathbf{C}_3 -type. Assume that, for any $x \in R$, the fixed point set $M^{\mathbf{K}_x}$ is a \mathbf{C}_2 -building. If \star holds, then $\mathcal{C}(M, \mathbf{G})$ is a \mathbf{C}_3 -building.*

Proof. If not, by Axiom (LL) there are two points $q \neq q' \in Q$ which are both incident to two different lines $x, x' \in R$. By \star we know that $\mathbf{G}_q \cap \mathbf{G}_{q'} \subset \mathbf{G}_{xq} \cap \mathbf{G}_{xq'}$ and $\mathbf{G}_q \cap \mathbf{G}_{q'} \subset \mathbf{G}_{x'q} \cap \mathbf{G}_{x'q'}$. Therefore, the configuration $\{xq, xq', x'q, x'q'\}$ is contained in the fixed point set $M^{\mathbf{G}_q \cap \mathbf{G}_{q'}}$. Note that \mathbf{K}_x is a subgroup of $\mathbf{G}_q \cap \mathbf{G}_{q'}$. Therefore, $M^{\mathbf{G}_q \cap \mathbf{G}_{q'}}$ is contained in $M^{\mathbf{K}_x}$. This implies that there is a length 4 circuit in the \mathbf{C}_2 building $M^{\mathbf{K}_x}$. A contradiction. \square

In our application we need the following more practical criterion.

LEMMA 5.4. *Let $\mathcal{C}(M, \mathbf{G})$ be a connected \mathbf{G} -homogeneous chamber system of \mathbf{C}_3 -type. Assume that, for any $x \in R$, the fixed point set $M^{\mathbf{K}_x}$ is a \mathbf{C}_2 -building. Then $\mathcal{C}(M, \mathbf{G})$ is a \mathbf{C}_3 -building, provided the following is true:*

(P) *For any $q \in Q$ incident to $x \in R$, if $\mathbf{L} \subset \mathbf{G}_q$ is a Lie subgroup so that $\mathbf{K}_x \subset \mathbf{L}$ but \mathbf{L} is not contained in \mathbf{G}_{xq} , then the normalizer $\mathbf{N}(\mathbf{K}_x) \cap \mathbf{L}$ is not contained in \mathbf{G}_{xq} .*

Proof. By the previous lemma it suffices to verify \star . Suppose \star is not true. Then there is an $x \in R$ and a pair of points $q \neq q'$ both incident to x such that $\mathbf{G}_q \cap \mathbf{G}_{q'}$ is not a subgroup of \mathbf{G}_{xq} . Let $\mathbf{L} = \mathbf{G}_q \cap \mathbf{G}_{q'}$. By Assumption (P), there is an $\alpha \in \mathbf{N}(\mathbf{K}_x) \cap \mathbf{L}$ so that $\alpha \notin \mathbf{G}_{xq}$. Note that $\mathbf{G}_x \cap \mathbf{N}(\mathbf{K}_x) \cap \mathbf{G}_q = \mathbf{G}_{xq} \cap \mathbf{N}(\mathbf{K}_x)$ since $M^{\mathbf{K}_x}$ is an \mathbf{C}_2 building. Therefore, $\alpha \notin \mathbf{G}_x$, and so there is a length 4 circuit $\{xq, xq', \alpha(x)q, \alpha(x)q'\}$ in the \mathbf{C}_2 building $M^{\mathbf{K}_x}$. A contradiction. \square

We will let \mathbf{G}_t^0 denote the kernel of the slice representation on T_t^\perp .

LEMMA 5.5. *Let M be a connected polar \mathbf{G} -manifold of \mathbf{C}_3 type. If the action is effective, then the kernel \mathbf{G}_t^0 is effective on the both slices T_q^\perp and T_r^\perp .*

Proof. Notice that \mathbf{G}_t^0 fixes all sections through t since \mathbf{G}_t^0 acts trivially on the slice T_t^\perp . It suffices to prove that $\mathbf{G}_t^0 \cap \mathbf{G}_q^0$ and $\mathbf{G}_t^0 \cap \mathbf{G}_r^0$ are both trivial. Since it is completely similar, we consider only $\mathbf{G}_t^0 \cap \mathbf{G}_q^0$.

Note that $\mathbf{G}_t^0 \cap \mathbf{G}_q^0$ is a normal subgroup of \mathbf{G}_t^0 acting trivially on both the slices T_t^\perp and T_q^\perp . Let $\tilde{\mathbf{G}}_t$ denote the quotient group $\mathbf{G}_t / \mathbf{G}_t^0$, and let $p : \mathbf{G}_t \rightarrow \tilde{\mathbf{G}}_t$ be the projection homomorphism. It is clear that \mathbf{G}_t^0 is contained in the subgroup $p^{-1}(\tilde{\mathbf{G}}_{t,0})$, where $\tilde{\mathbf{G}}_{t,0}$ is the identity component of $\tilde{\mathbf{G}}_t$. Therefore, without loss of generality, we may assume that $\tilde{\mathbf{G}}_t$ is a connected Lie group.

Let us first prove that the claim in the case that both \mathbf{G}_t and \mathbf{G}_q are connected. Under this additional assumption, $\mathbf{G}_t = \tilde{\mathbf{G}}_t^0 \times_z \tilde{\mathbf{G}}_t$, where z is a finite subgroup in the center of both $\tilde{\mathbf{G}}_t^0$ and $\tilde{\mathbf{G}}_t$. Therefore, a normal subgroup of the factor \mathbf{G}_t^0 must be also normal in \mathbf{G}_t . Hence $\mathbf{G}_t^0 \cap \mathbf{G}_q^0$ is a normal subgroup of \mathbf{G}_t . Similarly, $\mathbf{G}_t^0 \cap \mathbf{G}_q^0$ is also a normal subgroup of \mathbf{G}_q , therefore, a normal subgroup of \mathbf{G} by the primitivity since \mathbf{G}_t and \mathbf{G}_q generate \mathbf{G} . It is clear that $\mathbf{G}_t^0 \cap \mathbf{G}_q^0$ is contained in \mathbf{H} . This implies that $\mathbf{G}_t^0 \cap \mathbf{G}_q^0$ acts trivially on M . The desired claim follows.

For the general case we need only to notice that a continuous homomorphism from the connected Lie group $\tilde{\mathbf{G}}_t$ to a finite group is always trivial. Therefore the conjugation by elements of

\bar{G}_t gives a trivial automorphism on the normal subgroup G_t^0 . Hence, the above product decomposition of G_t still holds true. The rest of the proof follows similarly. \square

By applying Lemma 5.4 we get that

THEOREM 5.6. *Let M be a compact simply connected polar G -manifold of type C_3 with section $\Sigma = \mathbb{S}^2$. Assume that the multiplicity triple is $(1, 1, k)$ where $k \geq 2$. If the slice representation at $q \in Q$ is the defining tensor representation of $\mathrm{SO}(2) \mathrm{SO}(k+2)$, then the chamber system $\mathcal{C}(M; G)$ is a building as long as $\mathcal{C}(M, G)$ is connected.*

Proof. We first claim that G_t is connected. It suffices to prove that G / G_t is simply connected by the homotopy exact sequence of $G_t \rightarrow G \rightarrow G / G_t$. By [Wie], the homotopy fiber P of the inclusion $G / G_t \subset M$ has the same fundamental group as the homotopy fiber of the inclusion $\mathrm{SO}(3) \mathrm{SO}(k+3) / \Delta(\mathrm{SO}(3)) \mathrm{SO}(k) \subset \mathbb{S}^{3k+8}$. Therefore, $\pi_1(P) = \{1\}$. By the homotopy exact sequence of $P \rightarrow G / G_t \subset M$ it follows that G_t is connected.

Note that $\bar{G}_t = \mathrm{SO}(3)$, and the principal isotropy group of \bar{G}_t on the slice is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Therefore, $\bar{G}_q = \mathrm{S}(\mathrm{O}(2) \mathrm{O}(k+2))$. By Lemma 5.5 it is easy to see that, $G_t = \mathrm{SO}(3) \mathrm{SO}(k)$, $G_q = \mathrm{S}(\mathrm{O}(2) \mathrm{O}(k+2))$ and $G_r = \mathrm{SO}(k+1) \mathrm{SO}(2) \cdot \mathbb{Z}_2$, where the principal isotropy group $H = \mathrm{SO}(k) \cdot \mathbb{Z}_2^2$,

Let $K := K_r$ be the group defined above. Then $K = \mathrm{SO}(k+1)$. We first notice that the assumption (P) in Lemma 5.4 is satisfied, hence it remains to prove that M^K with the cohomogeneity 1 action by $N_0(K)$ gives rise to a C_2 -building where $N_0(K)$ is the identity component of $N(K)$. It is easy to see that the multiplicities for the $N(K)$ -action on M^K is $(1, 1)$. We will prove that M^K is equivariantly diffeomorphic to the linear tensor action of $\mathrm{SO}(2) \mathrm{SO}(3)$ on $\mathbb{S}^5 \subset \mathbb{R}^2 \otimes \mathbb{R}^3$, which implies the desired result.

For proving that, let us consider the fixed point set $M^{\mathrm{SO}(k)}$ of the kernel $\mathrm{SO}(k) \triangleleft G_t$, with the action by its normalizer $N(\mathrm{SO}(k))$. This is once again a polar fashion with the same section Σ , and the same orbit space. Notice, however, that the orbit type of the face ℓ_t is exceptional whose quotient by its principal isotropy group is \mathbb{Z}_2 . This is because that, $N(\mathrm{SO}(k))$ may not be connected, and $M^{\mathrm{SO}(k)}$ may not be simply connected.

Let $N_0(\mathrm{SO}(k))$ denote the identity component of $N(\mathrm{SO}(k))$. By lifting the action of $N_0(\mathrm{SO}(k))$ to the universal cover of $M^{\mathrm{SO}(k)}$, we see that the fundamental chamber is the union of two chambers C and $w_{\ell_t}(C)$ along the face ℓ_t in the section $\tilde{\Sigma}$, which is an A_3 geometry. By the previous section we know that any A_3 geometry is a building and hence a topological building, therefore $N_0(\mathrm{SO}(k))$ is $\mathrm{SO}(3) \mathrm{SO}(3)$ modulo the kernel whose action on the universal cover of $M^{\mathrm{SO}(k)}$ is equivariantly diffeomorphic to \mathbb{S}^8 with the linear action induced from the tensor representation on $\mathbb{R}^3 \otimes \mathbb{R}^3$. Since the section Σ is a sphere, the projective space \mathbb{RP}^8 can not appear. Hence $M^{\mathrm{SO}(k)} = \mathbb{S}^8$.

Note that $M^K \subset M^{\mathrm{SO}(k)}$ is a totally geodesic submanifold, which is included in the fixed point set of $\mathrm{SO}(k) \cdot \mathbb{Z}_2$, the normalizer of $\mathrm{SO}(k)$ in $\mathrm{SO}(k+1)$, which is the five sphere \mathbb{S}^5 with the linear action of $\mathrm{SO}(2) \mathrm{SO}(3)$ modulo kernel. It is clear that M^K has dimension at least 5 since the Weyl group for the cohomogeneity one action is D_4 . Therefore, $(M^K, N_0(K)) = (\mathbb{S}^5, \mathrm{SO}(2) \mathrm{SO}(3)K)$ up to equivariant diffeomorphism. This proves the desired result. \square

COROLLARY 5.7. *Let M be as in the above, but the section Σ is either \mathbb{S}^2 or \mathbb{RP}^2 . If $k \geq 2$ is even, then the chamber system $\mathcal{C}(M; G)$ is a building as long as $\mathcal{C}(M, G)$ is connected.*

Proof. We argue by induction. If $k = 2$, the fixed point set $M^{\mathrm{SO}(2)}$ is orientable since its normal bundle admits a complex structure. Hence, the above proof implies that $M^{\mathrm{SO}(2)}$ must be \mathbb{S}^8 . For larger even k , we can consider the block subgroup $\mathrm{SO}(2)$ in $\mathrm{SO}(k)$, it reduces in $\frac{k}{2}$ steps to $M^{\mathrm{SO}(k)}$, which is therefore orientable. The desired result follows. \square

We conclude this section with an equivalent description of covering of homogeneous chamber system, which plays an important role in the next section.

As noticed in Ronan [Ro], a homogeneous \mathbf{C}_3 type chamber system \mathcal{C} can be characterized by the isotropy group data:

- Let H be the principal isotropy group, and let G_{ℓ_i} , $i = q, r, t$, be the face isotropy groups. The chamber system \mathcal{C} is the left coset G / H (the principal orbit) with the following adjacency relation: two chambers gH and $g'H$ are i -adjacent if and only if $gG_{\ell_i} = g'G_{\ell_i}$.

Let \mathcal{C} be a chamber system of type \mathbf{C}_3 and let π be a group acting freely on \mathcal{C} preserving types. Then the quotient \mathcal{C}/π is also a chamber system of type \mathbf{C}_3 and the projection $\mathcal{C} \rightarrow \mathcal{C}/\pi$ is by definition a covering if it maps a rank 2 residue isomorphically onto a rank 2 residue.

For a homogeneous chamber system \mathcal{C} as above we have in particular the following:

- Let \mathcal{C} be a connected homogeneous \mathbf{C}_3 chamber system of cosets G / H with face isotropy groups G_{ℓ_i} , $i = q, r, t$. Let $\hat{G} = G \cdot \pi$ be a direct product. Let $\hat{H} \subset \hat{G}_{\ell_i} \subset \hat{G}_j$ be subgroups of $G \cdot \pi$ which project isomorphically onto $H \subset G_{\ell_i} \subset G_j$ in G under $G \cdot \pi \rightarrow G$, where G_j is the isotropy group of one of the vertices of ℓ_i . Then $G \cdot \pi / \hat{H}$ together with the adjacency relation defined by the face isotropy groups \hat{G}_{ℓ_i} , $i = q, r, t$, is a homogeneous \mathbf{C}_3 chamber system \mathcal{C}' covering \mathcal{C} , by the induced free π action.

We remark that the covering $G \cdot \pi / \hat{H}$ above is connected if and only if the face isotropy groups \hat{G}_{ℓ_i} , $i = q, r, t$, generates $G \cdot \pi$ (cf. Exercise 1 on p.7 in [Ro]). Moreover, \hat{G}_j projects isomorphically onto G_j if and only if the projection map between the chamber systems is an isomorphism on every rank 2 residue.

6. HOMOGENOUS \mathbf{C}_3 GEOMETRIES IN POSITIVE CURVATURE

The main result of this section is the following

THEOREM 6.1. *Let M be a compact, simply connected positively curved polar G -manifold of type \mathbf{C}_3 . Then M is G -equivariantly diffeomorphic to a rank one symmetric space with an isometric polar action.*

For the sake of simplicity, we will assume throughout this section that M is as in Theorem 6.1. Our proof of Theorem 6.1 will imply Theorem C for \mathbf{C}_3 type polar actions, namely, the chamber system $\mathcal{C}(M; G)$ is either a building or covered by a building, except in the case that M is G -diffeomorphic to the Cayley plane with an isometric action where $G = \mathrm{SU}(3) \mathrm{SU}(3)$ up to finite kernel.

Our strategy in the proof is to use Lemma 5.4, except for a few exceptional cases (including the Cayley plane example), that will be handled individually. We remark that, the local data for the isotropy groups can be obtained by using Lemma 5.5 and the slice representations. The difficulty is to what extent the data determine the group G .

Although many arguments are curvature free, we will use positive curvature if it allows us to simplify and shorten arguments. We start with a few frequently used results in the proof on the geometry of positive curvature, adapting them to our polar \mathbf{G} -action.

By applying Wilking's Isotropy Representation Lemma 3.1 in [Wi2] to our action we have the following important key lemma:

LEMMA 6.2 (Wilking). *For each normal simple group factor $K_i \triangleleft G_{\ell_i}$ where $i \in \{q, r, t\}$, every irreducible isotropy subrepresentation of \mathbf{G}/K_i is isomorphic to a standard defining representation of K_i , which is hence transitive on the unit sphere.*

By the well-known result of Berger for the zeros of Killing field we get immediately that

LEMMA 6.3. *The rank $rk(\mathbf{G}) \leq rk(\mathbf{G}_q) + 1$. Moreover, $rk(\mathbf{G}) = rk(\mathbf{G}_q)$ if the dimension M is even.*

By the classification of cohomogeneity one actions in [GWZ] it follows that

LEMMA 6.4. *For a cohomogeneity one action on a simply connected positively curved manifold, if multiplicity pair is not $(1, 1)$, $(1, 3)$ or $(1, 7)$, then the action is an isometric action on a rank one symmetric space up to equivariant diffeomorphism.*

We will divide the proof of Theorem 6.1 according to $d = 1, 2, 4$ using a uniform approach based on Lemma 5.4, except a few exceptional low dimensional cases, which will be divided into three parts.

THEOREM 6.5. *Assume that the multiplicity triple is $(1, 1, k)$ where $k \geq 2$. If the slice representation at $q \in Q$ is the defining tensor representation of $\mathrm{SO}(2) \mathrm{SO}(k+2)$, then the chamber system $\mathcal{C}(M; \mathbf{G})$ is a building.*

Proof. By Corollary 5.7 it suffices to prove the result for odd integer $k \geq 3$. Recall that $G_t = \mathrm{SO}(3) \mathrm{SO}(k)$ is connected, and $G_q = \mathrm{S}(\mathrm{O}(2) \mathrm{O}(k+2))$. Let $\mathrm{SO}(2) \triangleleft G_q$ be the normal subgroup. Consider the fixed point set $M^{\mathrm{SO}(2)}$, which is a homogeneous space a transitive action by the normalizer since the normal subgroup does not belong to any face isotropy groups up to conjugation. On the other hand, notice that $M^{\mathrm{SO}(2)}$ is positively curved of dimension at least 2, since $M^{\mathrm{SO}(2)} \cap M^{\mathrm{SO}(k)}$ is either \mathbb{S}^2 or \mathbb{RP}^2 . This implies that $M^{\mathrm{SO}(2)}$ is either \mathbb{S}^{k+2} or \mathbb{RP}^{k+2} , since the isotropy group for the transitive action is G_q . For the latter case, there is a subgroup $\mathrm{SO}(2) \mathrm{SO}(k+3) \subset \mathbf{G}$ acting transitively on \mathbb{RP}^{k+2} with isotropy group $\mathrm{SO}(2) \mathrm{O}(k+2)$, hence $\mathrm{SO}(2) \times \mathrm{O}(k+2) \subset G_q$. A contradiction. Therefore, $M^{\mathrm{SO}(2)} = \mathbb{S}^{k+2}$. In particular, $M^{\mathrm{SO}(2)} \cap M^{\mathrm{SO}(k)} = \mathbb{S}^2 \subset M^{\mathrm{SO}(k)}$. This implies readily that $\Sigma = \mathbb{S}^2$. By Theorem 5.6 the desired result follows. \square

Next we move to the case where $d = 2$. We start with the following key

LEMMA 6.6. *If the multiplicity triple is $(2, 2, 1)$, then the chamber system $\mathcal{C}(M; \mathbf{G})$ is covered by a building.*

Proof. Notice that $\bar{G}_t = \mathrm{SU}(3)/\mathbb{Z}_3$, $\bar{G}_q = \mathrm{S}(\mathrm{U}(2)\mathrm{U}(2))/\mathbb{Z}_2$. The principal isotropy group $H \cap \bar{G}_t = \mathbb{T}^2$. The slice representation of \bar{G}_q on the slice T_q^\perp is the same as the tensor representation of $\mathrm{SO}(2)\mathrm{SO}(4)$ on $\mathbb{R}^2 \otimes \mathbb{R}^4$ (modulo kernel). By Lemma 5.5 it is easy to get that, $G_t = \mathrm{SU}(3)/\mathbb{Z}_3$, $G_q = \mathrm{U}(2)\mathrm{U}(2) \cong \mathrm{SO}(2)\mathrm{SO}(4) \cdot \mathbb{S}^1$ where \mathbb{S}^1 is in the kernel in the slice representation, $G_r = \mathbb{S}^1 \cdot \mathrm{U}(2)$ where the $\mathrm{U}(2)$ factor in G_r is the face isotropy group of G_{ℓ_q} . Regarded as a subgroup of G_q , the isotropy group $G_{\ell_t} = \Delta(\mathrm{SO}(2))\mathrm{SO}(2) \cdot \mathbb{S}^1$.

Let $K = \Delta(\mathrm{SO}(2))$. It is easy to see that Assumption (P) is fulfilled. Consider similarly the fixed point set M^K with its normalizer $N := N(K)$ action. It is easy to see that this is a cohomogeneity one action with multiplicity pair $(2, 1)$. By [GWZ] (cf. Lemma 6.4) we get easily that M^K is equivariantly diffeomorphic to either \mathbb{S}^7 , a lens space $\mathbb{S}^7/\mathbb{Z}_p$ or \mathbb{CP}^3 associated to the linear action of $\mathrm{SO}(2)\mathrm{SO}(4)$ or $\mathrm{SO}(4)$ on $\mathbb{R}^2 \otimes \mathbb{R}^4$ modulo kernel.

If $M^K = \mathbb{S}^7$, by Lemma 5.4 it follows immediately that $\mathcal{C}(M, G)$ is a building.

For the latter two cases, we now construct a free \mathbb{S}^1 covering as follows:

Consider the subgroups triple $(\mathbb{S}^1 \cdot G_t, \mathbb{S}^1 \cdot G_r, \mathbb{S}^1 \cdot G_q)$ in $\mathbb{S}^1 \cdot G$. We want to construct a *compatible subgroups triple* $(\hat{G}_t, \hat{G}_q, \hat{G}_r)$ in $\mathbb{S}^1 \cdot G$ which projects isomorphically onto (G_t, G_q, G_r) in G and are compatible on common faces. This can be described as follows:

Let $\hat{G}_t = \{1\} \cdot G_t$, and let \hat{G}_q be the graph of the homomorphism $\mathrm{U}(2)\mathrm{U}(2) \rightarrow \mathbb{S}^1$ defined by sending (A, B) to $\det(A)\det(B)^{-1}$; and similarly, let \hat{G}_r be the graph of the projection homomorphism $G_r = \mathbb{S}^1 \cdot \mathrm{U}(2) \rightarrow \mathbb{S}^1$.

It is easy to check that this defines a free \mathbb{S}^1 -covering $\mathcal{C}(\widetilde{M}; G)$ of $\mathcal{C}(M, G)$, considered as cosets $\mathbb{S}^1 \cdot G/H$ with the adjacency relations using the face isotropy groups $(\hat{G}_t, \hat{G}_q, \hat{G}_r)$. We remark that the geometric realization of $\mathcal{C}(\widetilde{M}; G)$ is a principal \mathbb{S}^1 -bundle P over M .

Let \hat{K} be the subgroup of \hat{G}_{ℓ_t} isomorphically projects to K . The fixed point set $P^{\hat{K}}$ is a principal \mathbb{S}^1 -bundle over M^K . Notice that, the normalizer $N(\hat{K})$ in $\mathbb{S}^1 \cdot G$ is $\mathbb{S}^1 \cdot N$. By construction the singular isotropy groups at \hat{q} and \hat{r} are respectively isomorphic to $G_q \cap N$ and $G_r \cap N$. Moreover, $P^{\hat{K}} \rightarrow M^K$ is the Hopf bundle if $M^K = \mathbb{CP}^3$, and the bundle $\mathbb{S}^1 \times_{\mathbb{Z}_p} \mathbb{S}^7 \rightarrow \mathbb{S}^7/\mathbb{Z}_p$ if $M^K = \mathbb{S}^7/\mathbb{Z}_p$. For the latter we remark that the covering $\mathcal{C}(\widetilde{M}; G)$ may not be connected as a chamber system. By passing to a connected component of $\mathcal{C}(\widetilde{M}; G)$ we get therefore a connected covering of $\mathcal{C}(M; G)$ such that the covering restricts to a \mathbb{C}_2 building covering M^K . Applying Lemma 5.4 to a connected component of $\mathcal{C}(\widetilde{M}; G)$ the desired result follows. \square

We remark that the tensor representation of $\mathrm{SU}(3)\mathrm{SU}(3)$ on $\mathbb{C}^3 \otimes \mathbb{C}^3$ is not polar, but it is polar on the projective space $\mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^3)$. We warn that in the above construction of the covering, it is necessary that both G_q and G_r have \mathbb{T}^2 factors, since the face isotropy groups $G_{\ell_r} \cong G_{\ell_q} \cong \mathrm{U}(2)$ which are subgroups in $G_t = \mathrm{SU}(3)$, hence a compatible homomorphism to \mathbb{S}^1 will be trivial on the face isotropy groups.

THEOREM 6.7. *If the multiplicity triple is $(2, 2, 2k + 1)$ where $k \geq 2$, then the chamber system $\mathcal{C}(M; G)$ is covered by a building.*

Proof. By the same argument as before we get that G_t is connected, and hence so is G_q . Therefore, G_t is either $\mathrm{SU}(3)\mathrm{SU}(k)$ or $\mathrm{S}(\mathrm{U}(3)\mathrm{U}(k))$, up to possibly a kernel \mathbb{Z}_3 in the center in $\mathrm{SU}(3)$, e.g., if k is divisible by 3.

For simplicity we consider the case that $G_t = \mathrm{SU}(3) \mathrm{SU}(k)$, then the principal isotropy group $H = \mathrm{T}^2 \mathrm{SU}(k)$. The case that G_t is not semisimple is easier and similar.

By Lemma 5.5 we get immediately that $G_q = \mathrm{U}(2) \mathrm{SU}(k+2)$ and $G_r = \mathrm{U}(2) \mathrm{SU}(k+1)$, where $\mathrm{SU}(k+1) \triangleleft G_{\ell_t}$ is a block subgroup of G_q .

Notice that the kernel K_x defined in Lemma 5.4 is $\mathrm{SU}(k+1) \cdot \mathrm{S}^1 \triangleleft G_{\ell_t}$. It is convenient to consider the fixed point set M^K of the normal factor $K := \mathrm{SU}(k+1)$ with the induced action by the normalizer $N := N(K)$ as in Lemma 5.4. It is easy to see that the assumption (P) in Lemma 5.4 is satisfied.

Observe that, the orbit space of the cohomogeneity one N -action is $\overline{r\bar{q}}$, and the two singular isotropy groups (dividing the kernel) are respectively $\mathrm{SU}(2) \cdot \mathrm{S}^1$ and $\mathrm{SU}(2) \cdot \mathrm{T}^2$, and the principal isotropy group is T^2 . Therefore, the multiplicities is $(2, 3)$. We now prove that the N -action is not reducible. If not, M^K is equivariantly diffeomorphic to $\mathbb{S}^2 * \mathbb{S}^3$ with the product action of $\mathrm{SU}(2) \mathrm{U}(2)$, then the normal subgroup $\mathrm{SU}(2) \triangleleft G_q$ is a normal factor in N . By the primitivity $G = \langle G_r, G_q \rangle = \langle N, G_q \rangle$ it follows that $\mathrm{SU}(2)$ is normal in G . This contradicts to

- Claim: Neither $\mathrm{SU}(2)$ nor $\mathrm{SU}(k+2)$ factor of G_q is normal in G .

Indeed, if the factor is normal, then the projection homomorphism $G \rightarrow \mathrm{SU}(2)$ (resp. $\mathrm{SU}(k+2)$) would be nontrivial on $\mathrm{SU}(3) \triangleleft G_t \subset G$ by restricting the homomorphism to $G_{\ell_r} \subset G_t$. Absurd.

Therefore, by [GWZ] (cf. Lemma 6.4) we get easily that

SUBLEMMA 6.8. *M^K is equivariantly diffeomorphic to one of \mathbb{S}^{11} , $\mathbb{S}^{11}/\mathbb{Z}_p$, \mathbb{CP}^5 or \mathbb{HP}^2 with a linear action of $\mathrm{SU}(2) \mathrm{U}(3)$, $\mathrm{SU}(2) \mathrm{U}(3)$ or $\mathrm{SU}(3) \cdot \mathrm{S}^1$ (modulo the kernel), associated to the linear action of $\mathrm{U}(2) \mathrm{SU}(3)$ (resp. $\mathrm{SU}(2) \mathrm{SU}(3)$) on $\mathbb{C}^2 \otimes \mathbb{C}^3$, or the linear action of $\mathrm{SU}(3) \cdot \mathrm{S}^1 \subset \mathrm{Sp}(3) \mathrm{Sp}(1)$ on \mathbb{H}^3 .*

By Lemma 6.2 every irreducible isotropy subrepresentation of K is the defining representation $\rho : \mathrm{SU}(k+1) \rightarrow \mathrm{Aut}(\mathbb{C}^{k+1})$. Since $K \subset \mathrm{SU}(k+2) \triangleleft G_q$ is a block subgroup, by table B in [GWZ] and the above claim it follows that

- If $k \geq 3$, there is a simple normal subgroup $L \triangleleft G$ isomorphic to $\mathrm{SU}(n)$ for some $n \geq k+3$ such that $\mathrm{SU}(k+2) \triangleleft G_q$ is a block matrix subgroup of L . If $k = 2$, the simple group L is possibly $\mathrm{SO}(n)$ or one of the exceptional Lie groups $F_4 \subset E_6 \subset E_7 \subset E_8$.

By Theorem 6.6 the normalizer $N(\mathrm{SU}(k))$ where $\mathrm{SU}(k) \triangleleft G_t$, is either $\mathrm{SU}(3) \mathrm{SU}(3)$ or $\mathrm{U}(3) \mathrm{SU}(3)$ modulo the kernel. This implies easily that, $L = \mathrm{SU}(k+3)$ if $k \geq 3$. Moreover, if $k = 2$, then $L = \mathrm{SU}(5)$ or $\mathrm{SO}(7)$. In all cases, the slice K -representation contains exactly 3 copies of ρ . Therefore, the codimension of M^K in M is $6(k+1)$. This implies that the codimension of $M^{\mathrm{SU}(k)}$ in M is $6k$. We can now use the connectedness lemma of Wilking, to get that $\pi_i(M) \cong \pi_i(M^{\mathrm{SU}(k)})$ for $i \leq 2$, by induction on k . Therefore, M^K is simply connected. Moreover, by combing with the sublemma above, $M^K = \mathbb{S}^{11}$ if $\dim(M)$ is odd, and $M^K = \mathbb{CP}^5$ if $\dim(M)$ is even. Therefore, $\mathcal{C}(M; G)$ is a building if $\dim(M)$ is odd. We remark that the quaternionic projective plane does not occur since $M^K \subset M^{\mathrm{SU}(k)}$, and the latter is known from Theorem 6.6.

It remains to prove that $\mathcal{C}(M; G)$ is covered by a building if $\dim(M)$ is even. In this case, by the above we know that $\pi_2(M) \cong \pi_2(M^{\mathrm{SU}(k)}) \cong \mathbb{Z}$. On the other hand, by using the fibration $P \rightarrow G/G_t \rightarrow M$ and [Wie] it follows that $\pi_2(M) \cong \pi_2(G/G_t)$, where P is the path space which has the same homotopy information as the model space. Therefore, $\pi_1(G_t) \cong \mathbb{Z}$. Hence

G_t is not semi-simple. Indeed, $G_t = \mathrm{SU}(3) \mathrm{U}(k)$, and the principal isotropy group $H = \mathrm{T}^2 \cdot \mathrm{U}(k)$. By Lemma 5.5 we get similarly as before that $G_q = \mathrm{U}(2) \mathrm{U}(k+2)$ and $G_r = \mathrm{U}(2) \mathrm{U}(k+1)$, where $\mathrm{U}(k+1) \triangleleft G_{\ell_t}$ is a block subgroup of G_q . We are now in the same situation as in the proof of Lemma 6.6, we can define a free S^1 covering as follows:

Let $\hat{G}_q \subset \mathrm{S}^1 \cdot G_q$ (resp. $\hat{G}_t \subset \mathrm{S}^1 \cdot G_t$ and $\hat{G}_r \subset \mathrm{S}^1 \cdot G_r$) be the graph of the homomorphism $G_q \rightarrow \mathrm{S}^1$ by sending $(A, B) \mapsto \det(A)^{-1} \det(B)$. This determines all faces isotropy groups of the covering, by the compatibility. As in the proof of Lemma 6.6, this defines a free S^1 covering of $\mathcal{C}(M; \mathbf{G})$ whose restriction on M^K is the Hopf fibration, the chamber system is a \mathbf{C}_2 building corresponding to the tensor representation of $\mathrm{S}(\mathrm{U}(2) \mathrm{U}(3))$ on $\mathbb{C}^2 \otimes \mathbb{C}^3$. By Lemma 5.4 the desired result follows. \square

We remark that the above proof does not work for $k = 1$ since $\mathrm{SU}(1) = 0$.

THEOREM 6.9. *If the multiplicity triple is $(4, 4, 4k + 3)$ where $k \geq 0$, then the chamber system $\mathcal{C}(M; \mathbf{G})$ is a building.*

Proof. As in the proof of Theorem 5.6 we get similarly that G_t is connected, and hence so is the principal isotropy group and all isotropy groups. Note that, $\bar{G}_t = \mathrm{Sp}(3)/I$, where $I = \{\pm 1\}$, and $\bar{G}_q = \mathrm{Sp}(2) \times_I \mathrm{Sp}(k+2)$. The principal isotropy group for the slice representation of \bar{G}_q is then $\mathrm{Sp}(1)^2 \times_I \mathrm{Sp}(k)$. Hence, by Lemma 5.5 we get that, $G_t = \mathrm{Sp}(3) \times_I \mathrm{Sp}(k)$, $G_q = (\mathrm{Sp}(1) \mathrm{Sp}(2)) \times_I \mathrm{Sp}(k+2)$, $G_r = (\mathrm{Sp}(2) \mathrm{Sp}(1)) \times_I \mathrm{Sp}(k+1)$. This implies that, the principal isotropy group $H = \mathrm{Sp}(1)^3 \times_I \mathrm{Sp}(k)$, and the face isotropy group $G_{\ell_t} = \mathrm{Sp}(1)^3 \times_I \mathrm{Sp}(k+1)$. Notice that the kernel K_x defined in Lemma 5.4 is $\mathrm{Sp}(k+1) \mathrm{Sp}(1) \triangleleft G_{\ell_t}$. It is more convenient to consider the fixed point set M^K of the normal factor $K := \mathrm{Sp}(k+1)$ with the induced action by the normalizer $N := N(K)$ as in Lemma 5.4. It is easily seen that the assumption (P) in Lemma 5.4 is satisfied.

Observe that, the orbit space of the cohomogeneity one N -action is $\overline{r\bar{q}}$, and the two singular isotropy groups (dividing the kernel) $\mathrm{Sp}(2) \times_I \mathrm{Sp}(1)$ and $\mathrm{Sp}(2) \times_I \mathrm{Sp}(1)^2$. Therefore, the multiplicities pair is $(4, 7)$. As in the proof of Theorem 6.7 the action is not reducible, and hence by Lemma 6.4 it follows that, the universal cover \tilde{M}^K together with the lifted action is equivariantly diffeomorphic to the unit sphere in $\mathbb{H}^2 \otimes \mathbb{H}^3$ with the defining tensor action of $\mathrm{Sp}(2) \times_I \mathrm{Sp}(3)$. Therefore, by Lemma 5.4 it suffices to prove that M^K is simply connected.

On the other hand, by Lemma 6.2 and table B in [GWZ] we get easily that:

- There is a simple normal subgroup $L \triangleleft G$ isomorphic to $\mathrm{Sp}(n)$ for some $n \geq k+2$ such that $\mathrm{Sp}(k+2) \triangleleft G_q$ is a block matrix subgroup of L .

Notice that $n \geq k+3$ since $\mathrm{Sp}(k+2) \triangleleft G_q$ is not normal in G . Since $\tilde{N} = \mathrm{Sp}(2) \mathrm{Sp}(3)$ modulo kernel, by the above G can not be a simple group, moreover, \tilde{G} contains $\mathrm{Sp}(3) \mathrm{Sp}(n)$, because $G_{\ell_r} \subset G_t$ is diagonally imbedded. Hence, $\tilde{G} = \mathrm{Sp}(3) \mathrm{Sp}(k+3)$ by the primitivity. Moreover, $G = \mathrm{Sp}(3) \times_I \mathrm{Sp}(k+3)$ since $\pi_1(G/G_t) \cong \pi_1(M) = \{1\}$. Hence, $N = \mathrm{Sp}(2) \times_I \mathrm{Sp}(3)$. Therefore $\pi_1(M^K) = \{1\}$ since $\pi_1(N(q)) = \{1\}$. \square

Proof of exceptional cases: I

Now we turn to the exceptional cases which do not fit in the infinite series, but where the building approach can be applied directly :

LEMMA 6.10. *If the multiplicity triple is $(1, 1, 7)$, then the slice representation at q can not be the tensor representation of $\mathrm{SO}(2) \mathrm{Spin}(7)$ on $\mathbb{R}^2 \otimes \mathbb{R}^8$.*

Proof. Notice that the identity component of the principal isotropy group H is $\mathrm{SU}(3)$. Consider the fixed point set $M^{\mathrm{SU}(3)}$ with the action of its normalizer $N := N(\mathrm{SU}(3))$. This is again a cohomogeneity 2 polar action with finite principal isotropy group. After dividing the kernel for the slice representation of N -action at q , and singular isotropy group $G_q \cap N$ is T^2 , while the face isotropy groups at ℓ_t and ℓ_r are respectively S^1 . This is impossible, since the orbit space of the slice representation at q of T^2 must have angle $\pi/2$, rather than $\pi/4$. \square

LEMMA 6.11. *If the multiplicity triple is $(4, 4, 1)$, then the chamber system $\mathcal{C}(M; G)$ is either a building or a free S^1 covering $\tilde{\mathcal{C}}(M, G)$ is a building, which corresponds to the isotropy representation of $\mathrm{SO}(12)/\mathrm{U}(6)$ on the sphere \mathbb{S}^{29} .*

Proof. Up to \mathbb{Z}_2 kernel, we may write down all the isotropy groups as before, $G_t = \mathrm{Sp}(3)$, $G_q = \mathrm{SO}(2) \mathrm{Spin}(6) \mathrm{Sp}(1)$, and $G_r = \Delta(\mathrm{SO}(2)) \mathrm{Spin}(4) \mathrm{Sp}(1)$. Note that $G_{\ell_t} = \Delta(\mathrm{SO}(2)) \mathrm{Sp}(1) \mathrm{Spin}(4)$.

Let $K = \Delta(\mathrm{SO}(2)) \mathrm{Sp}(1) \triangleleft G_{\ell_t}$. Consider the fixed point set M^K . We remark that the factor $\mathrm{Sp}(1)$ is a normal subgroup in $\mathrm{Spin}(4) \subset G_{\ell_t}$. Let $N := N(K)$. The N -action on M^K is cohomogeneity 1 with principal isotropy group (dividing the kernel) $\mathrm{Sp}(1) \mathrm{Sp}(1)$, and two singular isotropy groups $\mathrm{Spin}(5)$ and $\mathrm{Sp}(1) \mathrm{Sp}(1) \cdot S^1$. Hence the multiplicity pair is $(4, 1)$. Once again, by Lemma 6.4 and the same argument as in Theorem 6.9 to exclude the reducible case, we know that the polar action on M^K is finite covered (or S^1 covered) by a linear polar action on the sphere, with linear model the tensor representation of $\mathrm{SO}(2) \mathrm{SO}(6)$ on $\mathbb{R}^2 \otimes \mathbb{R}^6$. It is easy to see that the assumption (P) in Lemma 5.4 is satisfied. By Lemma 5.4 $\mathcal{C}(M, G)$ is a building if M^K is a sphere.

If $M^K = \mathbb{CP}^5$ or a lens space $\mathbb{S}^{11}/\mathbb{Z}_p$, we similarly define a free S^1 covering chamber system $\tilde{\mathcal{C}}(M, G)$ as follows:

Let $(\hat{G}_t, \hat{G}_q, \hat{G}_r)$ be subgroups of $S^1 \cdot G$ such that $\hat{G}_t = \{1\} \cdot G_t$, and $\hat{G}_q \cong G_q$ and $\hat{G}_r \cong G_r$ are respectively the graphs of the projection homomorphisms $G_q \rightarrow S^1$ and $G_r \rightarrow S^1$. It is easy to check that it defines a free S^1 -covering of $\mathcal{C}(M, G)$, considered as cosets $S^1 \cdot G / H$ with the adjacency relations as above, whose restriction on the subcomplex M^K is either the Hopf fibration or the bundle $S^1 \times_{\mathbb{Z}_p} \mathbb{S}^{11} \rightarrow \mathbb{S}^{11}/\mathbb{Z}_p$. Therefore, once again by Lemma 5.4 $\mathcal{C}(M, G)$ is covered by a building. The desired result follows. \square

LEMMA 6.12. *If the multiplicity triple is $(8, 8, 1)$, then $\mathcal{C}(M, G)$ or its S^1 free covering is a building, with linear model the polar representation of $E_6 \times S^1$ on $\mathbb{S}^{53} \subset \mathbb{R}^{54}$.*

Proof. Up to a finite kernel, we may assume that $G_t = F_4$, $G_q = \mathrm{SO}(2) \mathrm{Spin}(10)$, and $G_r = \Delta(\mathrm{SO}(2)) \mathrm{Spin}(9)$. Notice that the principal isotropy group $H = \mathrm{Spin}(8)$ and $G_{\ell_t} = \Delta(\mathrm{SO}(2)) \mathrm{Spin}(8) \subset G_q$.

Let $K = \Delta(\mathrm{SO}(2))$. Consider the fixed point set M^K with the action of $N := N(K)$. This is a cohomogeneity 1 action with multiplicity pair $(8, 1)$. Therefore, by Lemma 6.4 we know that $M^{\Delta(\mathrm{SO}(2))}$ is finitely covered or S^1 covered by \mathbb{S}^{19} with the linear action of $\mathrm{SO}(2) \mathrm{SO}(10)$. It is easily seen that the assumption (P) in Lemma 5.4 is satisfied. We are now in a similar situation as before, notice that both G_{ℓ_r} and G_{ℓ_q} are simple group, its homomorphism image in S^1 is

always zero. We can define a free S^1 covering exactly as in Theorem 6.11, by taking \hat{G}_q, \hat{G}_r being the graphs of the projection homomorphisms $G_q \rightarrow S^1$, and $G_r \rightarrow S^1$. The desired result follows. \square

Proof of exceptional cases: II

Now we turn to the exceptional cases, where sometimes building can not be applied, or other methods need to be introduced.

We start with the only remaining multiplicity triple where $d = 4$.

LEMMA 6.13. *If the multiplicity triple is $(4, 4, 5)$, then the chamber system $\mathcal{C}(M; G)$ is either a building or a free S^1 covering $\mathcal{C}(M, G)$ is a building, which corresponds to the isotropy representation of $SO(14)/U(7)$ on the sphere S^{41} .*

Proof. Notice that the slice representation at q is the exterior representation of $SU(5)$ or respectively $U(5)$ on \mathbb{R}^{20} . As before we get similarly that G_t and all isotropy groups are connected. Moreover, by Proposition 5.5, if G_t is semisimple, then, up to finite kernel, $G_t = Sp(3)$, $G_r = Sp(2)SU(3)$, $G_q = SU(5)Sp(1)$ and $G_{\ell_t} = SU(3)Sp(1)^2$, where $Sp(1) \triangleleft G_q$ is in the kernel of the slice representation at q . If G_t is not semisimple, then G_t contains exactly an S^1 factor, so is all other isotropy groups.

We claim that $G = SU(7)$ or $U(7)$, according to whether there is an S^1 factor in G_t and the parity of the dimension. Since the proof is similar, we will prove in the case that G_t is semisimple.

By Lemma 6.2 the isotropy representation of $G/Sp(2)$ and $G/SU(3)$ are both spherical. By comparing the table B in [GWZ] it follows that, the simple group factor (normal subgroup) of G containing $SU(5) \subset G_q$ must be $SU(n)$, for some $n \geq 5$. In particular, if G is a simple group, then $G = SU(n)$, and $n \geq 7$ because $G_q = SU(5)Sp(1) \subset G$. Hence $G = SU(7)$ since the rank of G is at most $6 = rk(G_q) + 1$ by the rank lemma.

If G is not a simple group, consider the projection of G to the factor $SU(n)$. It is clear that $G_t = Sp(3)$ has nontrivial image in $SU(n)$ hence a subgroup therein, so is G_q . Therefore, by the above G contains $SU(7)$ as a normal subgroup with quotient a rank 1 group. It is easy to see that $G = U(7)$, moreover, the center S^1 acts freely on M .

In both cases, one can, either use Lemma 5.4 to the reduction $M^{SU(3)}$ to prove the desired result, or check by hand that the isotropy group data are the same as the linear model of the action on S^{41} (resp. \mathbb{CP}^{20}), and hence the desired result. \square

We remark that in the next case, the reduction will give a reducible cohomogeneity one action, but it is still C_2 when an additional \mathbb{Z}_2 action is introduced.

LEMMA 6.14. *If the multiplicity triple is $(2, 2, 2)$, then $\mathcal{C}(M, G)$ is a building, with linear models the adjoint polar representations of $SO(7)$ or $Sp(3)$ on S^{20} .*

Proof. As before we get similarly that, possibly an S^1 factor need to be dropped for all isotropy groups if M is odd dimensional, we have that $G_t = U(3)$, $G_q = SO(5)SO(2)$, and $G_r = SO(3)U(2)$. The slice representation at q is the adjoint representation of $SO(5)$ on \mathbb{R}^{10} (cf. [GWZ]). Hence $G_{\ell_t} = SO(3)SO(2)SO(2) \subset G_q$.

Let $K = \mathrm{SO}(3) \triangleleft G_{\ell_t}$. Consider the fixed point set M^K with the cohomogeneity 1 action by its normalizer $N =: N(K)$. For such an N -action, notice that the point q is exceptional orbit type with isotropy group $\mathrm{O}(2)$, by dividing the kernel. The action of the identity connected component N_0 on M^K has $r'r$ as the orbit space of length $\pi/2$, where r' is a reflection image of r along the geodesic rq . Moreover, the multiplicities are $(2, 2)$. Therefore, the N_0 -action on M^K is reducible. This implies readily that M^K is either \mathbb{S}^3 , \mathbb{S}^5 (resp. \mathbb{RP}^5), with the linear action of $\tilde{N}_0 = \mathbb{S}^3$ having two fixed points on \mathbb{S}^3 or the join action of $\tilde{N}_0 = \mathbb{S}^3 \cdot \mathbb{S}^3$ on \mathbb{S}^5 (resp. \mathbb{RP}^5). The full group $N = N_0 \cdot \mathbb{Z}_2$, where \mathbb{Z}_2 interchanges the two singular orbits. Here we remark that M^K must be simply connected, since otherwise a singular orbit is \mathbb{RP}^2 , which is the fixed point set of $\mathrm{SO}(3) \mathbb{S}^3 \triangleleft G_r$ in the orbit G / G_r , hence the isotropy group G_r is not connected, a contradiction.

We now prove that M^K can not be \mathbb{S}^3 . If not, by the above we get that, all the isotropy groups G_q, G_r, G_t are rank 2 groups without the \mathbb{S}^1 factor. Since the diameter is realized by rr' , hence the section Σ must be \mathbb{RP}^2 . Consider the circle subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(3) = K$ in the principal isotropy group. The fixed point set $M^{\mathrm{SO}(2)}$ with the normalizer action is polar with Σ as a section again, but it is reducible. It is easy to see that the fundamental chamber for this action is qrq' with multiplicities $2, 2$ for the two right angle faces, where q' is the reflection image of q . Therefore, the slice representation at q' for the action of $N(\mathrm{SO}(2))$ is the adjoint representation of $\mathrm{SO}(5)$ on \mathbb{R}^{10} . Moreover, q' is a fixed point. Therefore, by the main result of the next section we get that $M^{\mathrm{SO}(2)} = \mathbb{RP}^{10}$, since the section is \mathbb{RP}^2 . However, $M^K \subset M^{\mathrm{SO}(2)}$ is \mathbb{S}^3 , a contradiction. This proves that $M^K = \mathbb{S}^5$.

Now we can apply Lemma 5.4. Notice that Assumption (P) in Lemma 5.4 is satisfied. Moreover, the linear $N = N_0 \cdot \mathbb{Z}_2$ action on \mathbb{S}^5 indeed satisfies the axiom (LL), though the N_0 action is reducible. The desired result follows. \square

Next we move to the remaining case with $d = 2$, the multiplicity triple is $(2, 2, 3)$. By SubLemma 6.8, the reduction M^K is either \mathbb{S}^{11} , $\mathbb{S}^{11}/\mathbb{Z}_p$, \mathbb{CP}^5 or \mathbb{HP}^2 . This corresponds to the following:

LEMMA 6.15. *Assume that the multiplicity triple is $(2, 2, 3)$. If M^K is not \mathbb{HP}^2 , then the chamber system $\mathcal{C}(M; G)$ is covered by a building, with linear model the complex tensor representation of $\mathrm{SU}(3) \mathrm{SU}(4)$.*

Proof. As in the proof of Theorem 6.7, up to a finite kernel, G_t is either $G_t = \mathrm{SU}(3)$ or $\mathrm{U}(3)$. If $G_t = \mathrm{U}(3)$, then $G_q = \mathrm{U}(2) \mathrm{U}(3)$ and $G_r = \mathrm{U}(2) \mathrm{U}(2)$. The desired covering chamber system follows as in the proof of Theorem 6.7 (compare also the proof of Lemma 6.6).

Now we consider the case that $G_t = \mathrm{SU}(3)$. Hence $G_q = \mathrm{U}(2) \mathrm{SU}(3)$ and $G_r = \mathrm{SU}(2) \mathrm{U}(2)$. By the rank lemma $\mathrm{rk}(G) = 5$ (resp. $\mathrm{rk}(G) = 4$) if $\dim(M)$ is odd (resp. even). We claim

- G is not a simple group.

Indeed, by Sublemma 6.8 $N(K) = \mathrm{SU}(2) \mathrm{SU}(2) \mathrm{SU}(3)$. If $\mathrm{rk}(G) = 4$, by Borel and de Siebenthal [BS] none of the rank 4 simple group contain $N(K)$ as a maximal rank subgroup. If $\mathrm{rk}(G) = 5$ and G is simple, by Lemma 6.2 and table B in [GWZ] it follows that $G = \mathrm{SU}(6)$ and $K = \mathrm{SU}(2) \subset \mathrm{SU}(3) \triangleleft G_q$ is a block subgroup. Hence $N(K)$ contains $\mathrm{SU}(4)$. A contradiction.

Therefore, we may write $G = L_1 \cdot L_2$, where L_1, L_2 are nontrivial Lie groups. Without loss of generality we assume that $\mathrm{SU}(3) \triangleleft G_q$ has a nontrivial projection image in the factor L_2 . By the primitivity it is easy to see that G_t is diagonally imbedded in $L_1 \cdot L_2$ and $\mathrm{SU}(3) \triangleleft G_q$ is a

subgroup of L_2 , since $G = \langle G_t, G_{\ell_t} \rangle = \langle G_t, K \rangle$. In particular, both L_1 and L_2 have rank at least 2 since the projections from G_t are almost imbeddings. Again by Lemma 6.2 and table B in [GWZ] it follows that a covering of G is $SU(3) SU(4)$. In particular, this implies that $\dim(M)$ is odd.

We can verify the desired result, either by using Lemma 5.4 to the reduction M^K to prove that $\mathcal{C}(M; G)$ is a building, or check by hand that the isotropy group data are the same as the linear model. \square

LEMMA 6.16. *Assume that the multiplicity triple is $(2, 2, 3)$. If $M^K = \mathbb{H}P^2$, then M is G -equivariantly diffeomorphic to the Cayley plane $\mathbb{C}aP^2$ with an isometric action where $G = SU(3) \times_{\mathbb{Z}_3} SU(3)$.*

Proof. Note first that $\dim(M)$ is even since the codimension of M^K is even. The isotropy groups data are the same as in the previous lemma. However, $N(K) = SU(3) \cdot S^1$ modulo kernel. It is easy to see that, up to the center \mathbb{Z}_3 , $G_r = U(2) SU(2) \subset N(K)$, $G_t = SU(3)$, and $G_q = U(2) SU(3)$. Therefore, by the rank lemma $\text{rk}(G) = 4$. Note that the singular orbit of the linear action $N(K)$ at q is $\mathbb{C}P^2$.

By analyzing the isotropy representation $\rho : G_q \rightarrow O(T_q G(q))$ we will prove

- (i) the restriction of ρ on $U(2) \triangleleft G_q$ is a direct sum of the irreducible standard defining representation of $U(2)$. In particular, it contains no trivial summand.
- (ii) the restriction of ρ on $SU(3) \triangleleft G_q$ is trivial on a positive dimensional summand.

Claim (ii) implies that the fixed point set $M^{SU(3)}$ has dimension at least 1, and hence is either S^2 or $\mathbb{C}P^2$ in $M^K \cap G(q) = \mathbb{C}P^2$. If the 2-sphere occur, the normalizer of $SU(3)$ contains $SU(3) SU(2) SU(2)$ and hence $N(K)$ contains $SU(2) U(2)K$. Impossible. Hence, $M^{SU(3)} = \mathbb{C}P^2$, and so $SU(3) \times_{\mathbb{Z}_3} SU(3) \subset G$ acting on $M^{SU(3)}$ with isotropy group G_q . By the primitivity it follows that $G = \langle G_q, N(K) \rangle = SU(3) \times_{\mathbb{Z}_3} SU(3)$. It is easy to see that all data, as subgroups of $SU(3) \times_{\mathbb{Z}_3} SU(3)$, coincide with the polar C_3 type isometric action of $SU(3) \times_{\mathbb{Z}_3} SU(3)$ on the Cayley plane, and hence the desired result.

We now prove claim (i). Notice that, by Lemma 6.2 every irreducible summand of the isotropy representation of $SU(2) \triangleleft G_q$ is the standard defining representation of $SU(2)$. If the fixed point set $M^{SU(2)}$ is not a point through q , hence a homogeneous space (by the normalizer) of even dimension with positive curvature. For the same reason as above $M^{SU(2)}$ can not be S^2 . Therefore, $M^{SU(2)}$ is either $\mathbb{C}P^3$ or S^6 , with a transitive action by $SU(4)$ or G_2 . In both cases, however, $M^{SU(2)} \cap M^K = S^2$. This contradicts to, the isotropy action of $U(2)$ on the singular orbit $N(K)(q) = \mathbb{C}P^2$ is the standard defining representation of $U(2)$. Similarly, we conclude that the center $S^1 \subset U(2)$ is the scalar multiplication in the isotropy representation space. The claim (i) follows, since by lemma 6.2 all irreducible summand of both $\Delta(SU(2)) \triangleleft G_{\ell_r}$ and $K \subset G_q$ are the defining representation.

Now we prove claim (ii) by contradiction. We notice that, by claim (i) the restriction $\rho|_{SU(3)}$, is both complex and quaternionic linear. This implies that ρ is the tensor representation of $SU(3)$ with the defining representation of $U(2)$. On the other hand, by Lemma 6.2 again, $\rho|_{SU(3)}$, when restricted to every $SU(2)$ subgroup is the defining representation. Therefore, it is easy to see that every irreducible summand of $\rho|_{SU(3)}$ is the standard complex defining representation as well. Recall that the slice representation is the tensor representation on $\mathbb{C}^2 \otimes \mathbb{C}^3$. Therefore, we

can choose an S^1 subgroup of $U(3) \subset G_q$ such that its fixed point subspace in the tangent space $T_q M$ has (real) dimension at least 16, and hence so is the fixed point set M^{S^1} .

To continue we consider the induced polar action of $N(S^1)$ on M^{S^1} . Notice that S^1 is conjugate to a subgroup in the principal isotropy group in G_q , hence Σ is also a section for the subaction. It is easy to see that the isotropy group $N(S^1) \cap G_q = U(2)U(2)$, and the fundamental chamber for this subaction has three right angles, $qq'q''$, with all vertices having type q . In particular the type t vertex is not the most singular orbit type. On the other hand, we may appeal the dual generation lemma which will be proved in the reducible section to get that, the opposite submanifold of $N(S^1)(q)$ is \mathbb{S}^4 , the base of the normal slice T_q^\perp . Similarly, for the other vertices q', q'' . Therefore, it is easy to see that q, q', q'' are fixed points of $N(S^1) = U(2)U(2)$, and $M^{S^1} = \mathbb{H}P^2$. A contradiction. \square

Proof of exceptional cases: III

Now we turn to the exceptional case where the multiplicity triple is $(1, 1, 1)$. This case is difficult and different for two reasons, all the face isotropy groups have no simple subgroup, and moreover, if any reduction to cohomogeneity one will give a multiplicity pair $(1, 1)$ subaction, for which [GWZ] does not apply, unless the dimension is even. On the other hand, because the group G is small since its rank is at most 3, we can prove it case by case.

For simplicity we sometimes use the same notations to denote Lie groups with the same Lie algebra. But the real group can be recognized from the context.

LEMMA 6.17. *If the multiplicity triple is $(1, 1, 1)$, then M is G -equivariantly diffeomorphic to one of the following spaces with linear polar actions of C_3 type:*

- $G = SO(3)SO(4)$ or $SO(3)SO(3)$, acting on $\mathbb{S}^{11} \subset \mathbb{R}^3 \otimes \mathbb{R}^4$ or its descended action on $\mathbb{H}P^2$ corresponding to the tensor representation;
- $G = U(3)$ or $SU(3)$, acting on \mathbb{S}^{11} or its descended action on $\mathbb{C}P^5$ corresponding to the isotropy representation of $Sp(3)/U(3)$.

Proof. By the rank lemma we know that the rank of G is at most 3, since $SO(2)SO(3) \triangleleft G_q$ with at most finite quotient. As before we may derive that G_t is $SO(3)$ because M is simply connected. Therefore, the principal isotropy group $H = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We will first prove that

- G is not a simple group of rank 3.

Assume not, we will derive a contradiction.

We start with an observation: if G is a rank 3 simple group, i.e., one of $SO(6) = SU(4)$, $SO(7)$ or $Sp(3)$ (up to center), then every order 2 subgroup \mathbb{Z}_2 has a normalizer containing a rank at least 2 semisimple subgroup.

We now consider a subgroup $\mathbb{Z}_2 \subset \Delta(SO(2)) \triangleleft G_{\ell_r}$ and its fixed point set $M^{\mathbb{Z}_2}$. The induced polar action is of cohomogeneity 2 by the normalizer $N := N(\mathbb{Z}_2)$. Let N_0 denote the identity connected component of N . Notice, the action is now reducible, since the orbit type $G(t)$ is no longer the most singular orbit of N_0 . Hence, the action of N_0 is reducible whose orbit space has two right angles along the opposite face of q . The isotropy group at q , i.e, $N_0 \cap G_q$ is either the identity component of G_q , e.g., $G_q = U(2) \supset O(2) = G_{\ell_r}$, or T^2 , e.g., $G_q = S(O(2)O(3)) \supset \Delta(O(2)) = G_{\ell_r}$. By the main result of section 8 it follows that, because the face isotropy groups are all S^1 up to finite extension, the orbit $N_0(q)$ is at most \mathbb{S}^1 , and $N_0 \cap G_q$ is orbit equivalent

to N_0 on the opposite face of q . Hence, N_0 is either T^3 or $T^2 \cdot \mathrm{SU}(2)$. Here we remark that dimension of M must be odd by the rank lemma. This contradicts to the above observation.

Next we claim that

- If G is a rank 2 group, then either $(M, G) = (\mathbb{CP}^5, \mathrm{SU}(3))$ or $(\mathbb{HP}^2, \mathrm{SO}(3) \mathrm{SO}(3))$ up to equivariant diffeomorphism.

We first notice that the dimension of M is even, since both G / G_q and G / G_r have positive Euler characteristics, and the slices have even dimensions. Consider the fixed point set M^K , where $K = \mathrm{SO}(2) \triangleleft G_{\ell_r}$. This is an oriented, even dimensional cohomogeneity one space with the action of $N(K)$ and the multiplicity pair $(1, 1)$. As before, by [GWZ] we know that M^K is equivariant diffeomorphic to \mathbb{CP}^2 with the linear action of $\mathrm{SO}(3)$, here we excluded the 4-sphere since the Weyl group is either D_2 or D_4 . Using the same argument in the proof of Theorem 6.16 it follows that $\mathrm{SO}(2) \triangleleft G_q$ is free on the unit tangent sphere \mathbb{S}_q . Therefore, the isotropy representation of $\mathrm{SO}(2) \mathrm{SO}(3) \triangleleft G_q$ is complex.

If $\mathrm{SO}(3) \triangleleft G_q$ has a fixed point on \mathbb{S}_q , then $M^{\mathrm{SO}(3)}$ is a positive and (even) dimensional homogeneous space, with isotropy group G_q . Hence, $M^{\mathrm{SO}(3)} = \mathbb{RP}^2$ since it is contained in the singular orbit in M^K . This implies that the universal covering $\tilde{G} = \mathrm{SU}(2) \mathrm{SU}(2)$. It is trivial to see that, if G is not a simple group, we also have that $\tilde{G} = \mathrm{SU}(2) \mathrm{SU}(2)$. One can easily check that all the isotropy group data must be the one as in the linear example, and hence the desired claim.

If $\mathrm{SO}(3)$ has no fixed point on \mathbb{S}_q . Since the isotropy representation of $\mathrm{SO}(2) \mathrm{SO}(3) \triangleleft G_q$ is a complex without trivial representation summand, this rules out both the simple groups $\mathrm{Sp}(2) = \mathrm{SO}(5)$ and G_2 . Therefore, $G = \mathrm{SU}(3)$ if G is a simple group of rank 2, and $G_r \subset \mathrm{O}(3)$ is imbedded as the 3×3 matrix subgroup, and all isotropy groups are same as in the linear example, and hence the desired claim.

Finally let us consider the remaining case that G is not a simple Lie group and $\mathrm{rk}(G) = 3$. We may write $G = L_1 \cdot L_2$, where L_1 is of rank 1 and L_2 is of rank 2.

If L_1 is free on M , then $L_1 = S^1$ or S^3 , and L_2 acts on M/L_1 in a polar fashion of C_3 type. Moreover, M/L_1 is of even dimension. Notice that the rank of L_2 is 2, by the above we see that M/L_1 is either \mathbb{CP}^5 or \mathbb{HP}^2 . Therefore, M is equivariant diffeomorphic to \mathbb{S}^{11} with the linear polar action of $\mathrm{U}(3)$ or $\mathrm{SO}(3) \mathrm{SO}(4)$.

Hence we may assume now that L_1 is not free on M . Then $L_1 = S^1$ or $L_1 = \mathrm{Sp}(1)$ (resp. $\mathrm{SO}(3)$) must have nontrivial intersection with the isotropy group G_q . We now prove that $L_1 = S^1$ can not occur. Otherwise, a cyclic subgroup $\mathbb{Z}_m \subset L_1$ whose fixed point set $M^{\mathbb{Z}_m}$ is the orbit $G(q)$ (since it is a normal subgroup in G and does not intersect with other isotropy groups). It is clear that \mathbb{Z}_m is a subgroup of $\mathrm{SO}(2) \triangleleft G_q$ (since \mathbb{Z}_m is normal in G). Notice that the normal subgroup $\Delta(\mathrm{SO}(2))$ of G_{ℓ_r} , is contained in both $G_t = \mathrm{SO}(3)$ and G_q , which projects trivially to $L_1 = S^1$. On the other hand, $\Delta(\mathbb{Z}_m) \subset \Delta(\mathrm{SO}(2)) \subset \mathrm{SO}(2) \mathrm{SO}(3)$ projects isomorphically into L_1 . A contradiction.

If $L_1 = \mathrm{SO}(3)$ (resp. $\mathrm{Sp}(1)$) and $\mathbb{Z}_m \subset L_1$ is in the isotropy group G_q . By the above we may assume that L_2 is semi-simple (otherwise there is a circle factor in the decomposition anyway). By looking at $\Delta(\mathrm{SO}(2)) \subset G_{\ell_r}$ it follows that $\mathrm{SO}(2) \triangleleft G_q$ projects nontrivially to L_1 . Moreover, $G_t = \mathrm{SO}(3) \subset G$ is diagonally imbedded, $\mathrm{SO}(3) \triangleleft G_q$ is contained in L_2 . Notice that the fixed point set $M^{\mathbb{Z}_m}$ is contained in the orbit $G(q)$. Therefore, the normalizer of \mathbb{Z}_m , $S^1 \cdot L_2$, acts transitively on $M^{\mathbb{Z}_m}$ with isotropy group $G_q \cap S^1 \cdot L_2 \subset \mathrm{SO}(2) \mathrm{SO}(3)$, which is a positively

curved homogeneous space. Therefore, $L_2 = \mathrm{SO}(4)$, $M^{\mathbb{Z}_m} = \mathbb{S}^3$, and hence $G = \mathrm{SO}(3) \mathrm{SO}(4)$. It is easy to verify, either by hand on all the isotropy group data, or by using Lemma 5.4, the desired result. \square

Finally, let us consider the last case where the multiplicity triple is $(1, 1, 5)$.

LEMMA 6.18. *Assume that the multiplicity triple is $(1, 1, 5)$. If the slice representation at q is the tensor representation of $\mathrm{SO}(2) G_2$ on $\mathbb{R}^2 \otimes \mathbb{R}^7$, then M is equivariantly diffeomorphic to \mathbb{S}^{23} induced from the tensor representation of $\mathrm{SO}(3) \mathrm{Spin}(7)$ on $\mathbb{R}^3 \otimes \mathbb{R}^8$.*

Proof. As before G_t is connected, $\tilde{G}_q = \mathrm{SO}(2) G_2$. Hence, it is easy to get that $G_t = \mathrm{SO}(3) \mathrm{SU}(2)$, $G_q = \mathrm{SO}(2) G_2 \cdot \mathbb{Z}_2$, and $G_r = \mathrm{SU}(3) \mathrm{O}(2)$. By the rank lemma $\mathrm{rk}(G) \leq 4$. We claim that $\mathrm{rk}(G) = 4$, since, if the rank were 3, then G_2 would be a normal subgroup of G , absurd similar to the claim in the proof of Theorem 6.7.

Therefore, $\dim(M)$ is odd by the rank lemma. Let $H = \mathrm{SU}(2)$ denote the identity connected component of the principal isotropy group. Consider the fixed point set M^H . The normalizer $N(H)$ action is polar with the same section Σ , and the same orbit space, but the multiplicity triple is $(1, 1, 1)$. Here we remark the dimension of the fixed point set is odd since the codimension is divisible by 4, by lemma 6.2. By appealing to Lemma 6.17 we see that the universal covering \tilde{M}^H is \mathbb{S}^{11} . Moreover, $N(H)$ is either $\mathrm{U}(3)$ or $\mathrm{SO}(3) \mathrm{SO}(4)$, modulo kernel.

Notice that $G_{\ell_i} = \mathrm{SU}(3) \cdot \mathbb{Z}_2^2$. By Lemma 6.2, the isotropy representation of $\mathrm{SU}(3) \subset G_2 \subset G$ is spherical. This implies that, G contains a normal simple Lie subgroup L , such that $G_2 \subset \mathrm{Spin}(7) \subset L$ is a block subgroup. Therefore, $L = \mathrm{Spin}(7)$ since $N(H)$ does not contain $\mathrm{Spin}(5)$. Because G_t is diagonally imbedded in $G = L_1 \cdot L$ where $L_1 = G/L$, we get immediately that $G \supset \mathrm{SO}(3) \mathrm{Spin}(7)$. Hence, by the primitivity, $G = \mathrm{SO}(3) \mathrm{Spin}(7)$. The desired result follows, either by comparing the isotropy group data with the linear model, or by using Lemma 5.3 via verifying that M^K is simply connected and hence the chamber system is a C_2 building, where $K = \mathrm{SU}(3)$. \square

We remark that our proof can be dramatically simplified if the following is true:

CONJECTURE. *If \mathcal{C} is a simply connected chamber system of C_3 type, then any connected C_2 subsystem is a building.*

Theorem 4.10 implies Theorem A in the introduction except when there are fixed points, or when the diagram either contains isolated nodes or is of type C_3 (see Theorem 3.12). In Section 4, all cases where the type of W is C_3 were discussed with direct methods and it was shown in Theorem 6.1 that the universal cover of the chamber complex $\mathcal{C}(M; G)$ is a building unless $M = \mathrm{CaP}^2$ and $G = \mathrm{SU}(3) \cdot \mathrm{SU}(3)$ acting in a polar fashion with cohomogeneity 2 (with finite kernel). Since, the Cayley plane is not the base of a fibration from a sphere, otherwise, the mapping cone will give a 24-dimensional manifold with only non-trivial homology at dimensions $8i$, for $i = 0, 1, 2, 3$. A contradiction to a well-known topological theorem (cf. [Ha]). The methods of proof of Theorem 4.10 allow us to conclude

PROPOSITION 6.19 (Not a Building). *The universal cover of the chamber system $\mathcal{C}(\mathrm{CaP}^2, G)$ associated with the polar action on CaP^2 by $G = \mathrm{SU}(3) \cdot \mathrm{SU}(3)$ is a simply connected chamber system of type C_3 which is not a building.*

The first known examples of simply connected chamber systems of type C_3 that are not buildings were discovered by Neumaier and later but independently by Aschbacher. To the best of our knowledge, Example 6.19 is new. It was independently discovered by Lytchak [Ly]. This intriguing example motivates the following interesting problems.

PROBLEM 6.20 (Cayley plane chamber system). *Let $\tilde{\mathcal{C}}$ denote the universal cover of the chamber system $\mathcal{C} := \mathcal{C}(\text{CaP}^2; \mathbf{G})$.*

- (1) *Is \mathcal{C} itself simply connected?*
- (2) *If \mathcal{C} is not simply connected, does the section \mathbb{RP}^2 lift to \mathbb{S}^2 in $\tilde{\mathcal{C}}$?*

Note that the Cayley plane example has multiplicity triple $(2, 2, 3)$. By the proof of Theorem 6.7 it seems that the chamber system $\mathcal{C}(\text{CaP}^2; \mathbf{G})$ can not have a covering so that the apartments are spheres. This seems to suggest that the chamber system of the Cayley plane example is simply connected.

Another fact we would like to point out at this stage is that, the action of $\text{SU}(3) \text{SU}(3)$ on the Cayley plane CaP^2 has exactly the same local data as the polar action of $\text{SU}(3) \text{SU}(4)$ on $\mathbb{S}^{23} \subset \mathbb{C}^3 \otimes \mathbb{C}^4$. Notice also the quotient $\text{SU}(3) \text{SU}(4) / \text{SU}(3) \text{SU}(3) = \mathbb{S}^7$. The latter action in some sense looks like a *fake* universal covering of \mathcal{C} but obvious not. \mathbb{S}^{23} is not a \mathbb{S}^7 bundle over CaP^2 !

7. REDUCTION INPUT AND FIXED POINT CASE

In the remaining two sections we will deal with reducible polar actions in positive curvature.

The key result in this section is a characterization of Hopf fibrations in our context, that also will play an essential role in the next section. As a corollary we obtain a classification when fixed points are present. To achieve the desired characterization we use our work on buildings and chamber systems from the previous sections and a description of all possible reducible polar representations. Such representations can be broken up into three types which we refer to as *standard*, *special* and *exceptional*. The precise definitions and a table exhibiting the exceptional ones due to Bergmann [Be] is provided in the appendix.

LEMMA 7.1 (Hopf fibration). *Let $(\mathbb{S}^n, \mathbf{G})$ be a polar representation without fixed points, and (B, \mathbf{G}) a simply connected closed polar space form of spherical type (frequently with a nontrivial rank one kernel). Suppose $p : \mathbb{S}^n \rightarrow B$ is a smooth, \mathbf{G} equivariant, chamber preserving map with $\mathbb{S}^n / \mathbf{G} = B / \mathbf{G}$. If for each $v \in \mathbb{S}^n$ the slice representations of \mathbf{G}_v and of $\mathbf{G}_{p(v)}$ are orbit equivalent, then p is either a diffeomorphism, or \mathbf{G} -equivalent to a Hopf fibration. Moreover, if $\dim B < n$ and B is a sphere the cohomogeneity is at most 1.*

Proof. Note that by assumption the chambers C in \mathbb{S}^n and B are spherical k -simplices, where k is the cohomogeneity of the actions, and p is surjective. Moreover, the assumption about slice representations imply that p is a submersion, and when restricted to a section Σ in \mathbb{S}^n is a cover of a section in B . This in particular proves our claim when $\dim B = n$, and from topology we already know that p is a fiber bundle with fiber \mathbb{S}^1 , \mathbb{S}^3 or \mathbb{S}^7 (all homogeneous in our case), where the latter can only happen when $n = 15$ (see [Ad], [Ha]).

Our proof for the case $\dim B < n$ is by induction on the number of isolated nodes in the diagram for the Coxeter matrix M associated to the Coxeter group W for the polar G representation. The induction anchor, i.e., when no isolated nodes are present, is divided into two cases, cohomogeneity $k = 1$ and $k > 1$.

- Cohomogeneity $k = 1$.

Since the cohomology of B is that of a projective space, this case can be derived from the classification of cohomogeneity one actions on spaces with rational cohomology of a projective space in [Uc] and [Iw1] [Iw2].

Alternatively, let H respectively K_{\pm} denote the principal, respectively singular isotropy groups of the G -representation on \mathbb{S}^n . Similarly, let H' and K'_{\pm} denote the corresponding data for the G -action on B . Observe that the fibers of p are $H' / H = K'_{\pm} / K_{\pm}$, which in turn is one of $\mathbb{S}^1, \mathbb{S}^3$ or \mathbb{S}^7 . Moreover, from our assumption on slice representations, $K_{\pm} / H = \mathbb{S}^{\ell_{\pm}} = K'_{\pm} / H'$.

In cohomogeneity one, no isolated nodes means that the G representation is irreducible. From the list of irreducible representations on odd spheres with cohomogeneity one given in table E in [GWZ] ($\ell_+ = 4k - 1$ in first row, not $4k + 1$) it is then immediate that only the fibers \mathbb{S}^1 and \mathbb{S}^3 can occur. In those cases $H \triangleleft H'$ is normal with quotient \mathbb{S}^1 or \mathbb{S}^3 , respectively, and it happens exactly when M has type C_2 . Using this information on G and its isotropy groups on \mathbb{S}^n and on B , it is a simple matter to see that B is G -equivariantly diffeomorphic to a complex, or quaternionic projective space, and that $p : \mathbb{S}^n \rightarrow B$ is G -equivariantly equivalent to a Hopf fibration.

- Cohomogeneity $k \geq 2$.

Notice that p induces a G equivariant surjective map between the chamber systems $\mathcal{C}(\mathbb{S}^n, G) \rightarrow \mathcal{C}(B, G)$ of type M . Since $\mathcal{C}(\mathbb{S}^n, G)$ is a building it is both connected and simply connected. In particular, $\mathcal{C}(B, G)$ is connected.

By our assumption on the slice representations it follows that p yields an isomorphism between all proper residues of $\mathcal{C}(\mathbb{S}^n, G)$ and $\mathcal{C}(B, G)$. In particular, $p : \mathcal{C}(\mathbb{S}^n, G) \rightarrow \mathcal{C}(B, G)$ is a covering map between chamber systems of type M , and hence $\mathcal{C}(\mathbb{S}^n, G)$ is the universal cover of $\mathcal{C}(B, G)$.

By construction of the chamber topology of the universal cover $\tilde{\mathcal{C}}(B, G) = \mathcal{C}(\mathbb{S}^n, G)$ in the previous section, it is apparent that it coincides with the topology on $\mathcal{C}(\mathbb{S}^n, G)$ defined using the Hausdorff metric on all compact subsets of \mathbb{S}^n . The corresponding thick topologies on $\mathcal{C}(B, G)$ and $\mathcal{C}(\mathbb{S}^n, G)$ yield the original topologies on B and \mathbb{S}^n respectively. Moreover, with this topology $\tilde{\mathcal{C}}(B, G) = \mathcal{C}(\mathbb{S}^n, G)$ is a topological building.

From Theorem 4.9 we also know that the fundamental group π of the cover $\mathcal{C}(\mathbb{S}^n, G) \rightarrow \mathcal{C}(B, G)$ is a compact subgroup of the topological automorphism group $\text{Aut}_{\text{top}}(\mathcal{C}(\mathbb{S}^n, G))$, and that there is an action by $\tilde{G} \subset \text{Aut}_{\text{top}}(\mathcal{C}(\mathbb{S}^n, G))$ covering the \bar{G} -action on \mathcal{C} , where \bar{G} is G mod its kernel on B , and \tilde{G} is an extension of \bar{G} by π . Moreover, the actions by $G \subset \tilde{G}$ on \mathbb{S}^n are orbit equivalent, and B is homeomorphic to \mathbb{S}^n / π .

Although in complete generality, we do not know much about the group $\text{Aut}_{\text{top}}(\mathcal{C}(\mathbb{S}^n, G))$, we claim that in our case when no isolated nodes are present, $\pi \subset \tilde{G}$ is either \mathbb{S}^1 or \mathbb{S}^3 acting freely on \mathbb{S}^n by the Hopf action.

Indeed, when the Coxeter diagram for M has no isolated nodes, $\text{Aut}_{\text{top}}(\mathcal{C}(\mathbb{S}^n, G))$ is a Lie group by [GKMW] (the rank is at least 3). Moreover, since its maximal compact subgroup acts

linearly on \mathbb{S}^n , the compact group π acts linearly and freely on \mathbb{S}^n , hence π is either trivial, \mathbb{S}^1 or \mathbb{S}^3 acting on \mathbb{S}^n by the Hopf action. Notice that G is either \bar{G} or \tilde{G} up to finite kernel.

This together with the irreducible case of cohomogeneity one establishes the induction anchor.

So assume, we have established our claim for any M with $\ell \geq 0$ isolated nodes, and consider an M with $\ell + 1 \geq 1$ isolated nodes. Say q_- is a vertex of a chamber simplex C corresponding to an isolated node of the diagram for M , and let Δ_+ denote the face opposite q_- . Then the spherical join $C = q_- * \Delta_+$, corresponds to a decomposition $V_- \oplus V_+$ for the linear representation of G on $\mathbb{S}^n = \mathbb{S}(V_- \oplus V_+) = \mathbb{S}_- * \mathbb{S}_+$, and G acts transitively on the sphere \mathbb{S}_- . Moreover, \mathbb{S}_- fibers over the corresponding G orbit $B_- \subset B$. From the classification of transitive actions on spheres, it is clear that the restriction $p_- : \mathbb{S}_- \rightarrow B_-$ is a Hopf fibration. Also, from our induction hypothesis, the same holds for the restriction $p_+ : \mathbb{S}_+ \rightarrow B_+$, where B_+ is the polar space form G submanifold, $G\Delta_+ \subset B$.

In the special case where the fiber is \mathbb{S}^7 it follows that $V_- \cong \mathbb{R}^8 \cong V_+$, and that B_- and B_+ are both fixed points of G . In particular, $G/K_{\pm} = \mathbb{S}^7$, $K'_{\pm} = G$, so $G/H' = \mathbb{S}^7$. Since also $K'_{\pm}/K_{\pm} = \mathbb{S}^7 = H/H'$, it follows from the list of transitive actions on the 7-sphere, that $G = \text{Spin}(8)$ with the exceptional reducible representation on \mathbb{R}^{16} with principal isotropy G_2 (cf. Table E in [GWZ]). Since B is equivariantly diffeomorphic to \mathbb{S}^8 with the suspension action by $\text{Spin}(8)$ our claim follows for this special case.

It remains to consider the cases where the fiber of p is either \mathbb{S}^1 or \mathbb{S}^3 . Since exceptional representations clearly do not support such fibrations (cf. Table 9.2), we conclude that the G representation on $V_- \oplus V_+$ is either standard or special.

- Suppose the polar G representation on $V_- \oplus V_+$ is standard:

By definition, we can write $G = G_- \cdot G_+ \cdot K$, where $K = \{1\}$, \mathbb{S}^1 or $\text{Sp}(1)$, and $G_- \cdot G_+$ acts componentwise on $V_- \oplus V_+$ orbit equivalent to the G -action. Since in particular p_{\pm} are G_{\pm} equivalent to Hopf fibrations $\mathbb{S}_{\pm} \rightarrow \mathbb{P}(V_{\pm})$ of the same type it follows that the linear sum polar representation action of $G_- \cdot G_+$ on $V_- \oplus V_+$ descends to the projective space $\mathbb{P}(V_- \oplus V_+)$. We claim that $p : \mathbb{S}(V_- \oplus V_+) \rightarrow B$ is $G_- \cdot G_+$ equivalent to the Hopf fibration $\mathbb{S}(V_- \oplus V_+) \rightarrow \mathbb{P}(V_- \oplus V_+)$. To see this we identify a chamber $C = q_- * \Delta_+$ in B with a chamber in $\mathbb{P}(V_- \oplus V_+)$ and use the chamber transitive $G_- \cdot G_+$ -actions to construct the desired $G_- \cdot G_+$ equivariant diffeomorphism between B and $\mathbb{P}(V_- \oplus V_+)$. It follows that the Hopf map $\mathbb{S}(V_- \oplus V_+) \rightarrow \mathbb{P}(V_- \oplus V_+)$ is G equivariant relative to the induced G action on $\mathbb{P}(V_- \oplus V_+)$ via B . In particular, the linear K action takes Hopf fibers to Hopf fibers and the induced action on $\mathbb{P}(V_- \oplus V_+)$ is isometric relative to the standard metric. Since it is orbit equivalent to the polar sub action by $G_- \cdot G_+$, it itself is polar.

- Suppose the polar G representation on $V_- \oplus V_+$ is special:

By Lemma 9.4 either

- (a) $G = G_- \cdot G_+ \cdot K$, where K (either \mathbb{S}^1 or $\text{Sp}(1)$) acts freely on $\mathbb{S}(V_-)$. Moreover, $G_- \cdot K$ is orbit equivalent to G_- on V_- , whereas $G_+ \cdot K$ is not orbit equivalent to G_+ on V_+ ;
- (b) $G = G_+ \cdot K$, where K is \mathbb{S}^1 or \mathbb{S}^3 acting transitively on $\mathbb{S}(V_-)$. Moreover, $G_+ \cdot K$ is orbit equivalent to G_+ on V_+ and K acts nontrivially on V_+ ;

In case (a) p_{\pm} is equivalent to a Hopf fibration $\mathbb{S}_{\pm} \rightarrow \mathbb{P}(V_{\pm})$ where K acts trivially on the base, i.e., as a sub action of the Hopf action on \mathbb{S}_{\pm} . In particular, the $G_- \cdot G_+$ action descends to a polar action on $\mathbb{P}(V_- \oplus V_+)$. Moreover, from the slice representation in B along B_+ we see that the $G_- \cdot G_+$ action on B is orbit equivalent to the G action. As above it follows that B

and $\mathbb{P}(V_- \oplus V_+)$ are $G_- \cdot G_+$ diffeomorphic, and hence p is G equivalent to the Hopf fibration $\mathbb{S}(V_- \oplus V_+) \rightarrow \mathbb{P}(V_- \oplus V_+)$.

In case (b), note that V_- is either \mathbb{R}^2 or \mathbb{R}^4 since the dimension of \mathbb{S}_- must be either 1 or 3. We claim that B_- is a point fixed point of G -action.

If not, then $K = \mathbb{S}^3$, $B_- = \mathbb{CP}^1$ and $p_{\mp} : \mathbb{S}(V_{\mp}) \rightarrow B_{\mp}$ are Hopf fibrations with fiber \mathbb{S}^1 . By [EH] we know that K acts trivially on every irreducible summand of V_+ of rank at least 2. Notice that, for every rank one summand of V_+ , since both $K \cdot G_+$ and G_+ act transitively on the sphere, it follows that K is either trivial or free on that sphere. On the other hand, the restriction of p on the sphere is a homogeneous Hopf \mathbb{S}^1 fibration over a complex projective space. Therefore, $K \cdot G_+$ commutes with the free \mathbb{S}^1 -action. This implies readily that K is trivial on V_+ , and so the G -representation is standard, a contradiction.

It is now easy to see that the cut locus of the fixed point B_- is B_+ and that the map $\mathbb{S}_{B_-} \rightarrow B_+$ taking a unit vector at B_- to the cut point of the corresponding geodesic satisfies the conditions of the Hopf lemma (see details below). Applying the induction hypothesis we deduce that B is G equivariantly diffeomorphic to the projective space $\mathbb{P}(V_- \oplus V_+)$. As in the standard case above this implies that $p : \mathbb{S}(V_- \oplus V_+) \rightarrow B$ is G -equivalent to the Hopf bundle $\mathbb{S}(V_- \oplus V_+) \rightarrow \mathbb{P}(V_- \oplus V_+)$. \square

We are now ready to prove the following

THEOREM 7.2. *Let M be a simply connected compact positively curved polar G manifold. If $M^G \neq \emptyset$ then (M, G) is equivariantly diffeomorphic to an isometric, polar action of G on a compact rank one symmetric space.*

Proof. We will see in particular that M is a sphere if and only if the section Σ is a sphere.

Let us first deal with the case where

- Σ is a k -sphere, $k \geq 2$:

Say $\Sigma = \mathbb{S}^{m+\ell+1}$, $F = \mathbb{S}^m$, $m \geq 0$ the common fixed point set of W and G , and $F' = \mathbb{S}^{\ell}$, $\ell \geq 1$, the W invariant “dual” sphere at distance $\pi/2$ to F in Σ . The smooth spherical join description $\Sigma = \mathbb{S}^m * \mathbb{S}^{\ell} = F * F'$ yields a decomposition of Σ as a union of the tubular neighborhoods of F and of F' . Applying G gives a smooth decomposition of M into the union of tubular neighborhoods of F and the G invariant manifold $GF' =: \hat{F} \subset M$. (In the metric chosen note that the cut locus of F is \hat{F} and vice versa, at distance $\pi/2$ from one another). It is clear from the slice representations that indeed \hat{F} is a polar G manifold with section F' , polar group W and $\hat{F}/G = F'/W$ a spherical ℓ -simplex. If $\mathbb{S}^0 = \{p_- \cup p_+\} \subset F$ is a pair of antipodal points in \mathbb{S}^m , it is clear that $G\mathbb{S}^{\ell+1} = G(\{p_- \cup p_+\} * F')$ is a G invariant polar submanifold $N \subset M$ with two isolated fixed points p_{\pm} , section $(\{p_- \cup p_+\} * F')$ and polar group W . From this it in particular follows that \hat{F} is equivariantly diffeomorphic to the unit sphere $\mathbb{S}^{\ell+1}$ of a fixed point, say p_- of G in N , and that N is equivariantly diffeomorphic to the suspension of this. Of course $\mathbb{S}^{\ell+1}$ is the normal sphere of F in M at say p_- , and a similar argument now shows that M is equivariantly diffeomorphic to $F * \mathbb{S}^{\ell+1} = \mathbb{S}^m * \mathbb{S}^{\ell+1}$ where G acts trivially on \mathbb{S}^m and acts as the differential of G on the normal sphere $\mathbb{S}^{\ell+1}$.

We now turn to the case where

- Σ is a projective k -space, $k \geq 2$:

Since Σ^* is a spherical k -simplex and $\text{Fix } G \subset \text{Fix } W$, we know it is contained in the vertices by Proposition 2.7.

Let $p \in F$ be such an isolated fixed point. From our description we know that p is the isolated fixed point of the reflection, r whose mirror corresponds to the chamber face Δ_0 opposite of p^* in $C = \Sigma^* = \Delta$. From the constant curvature model, it is clear that the mirror Λ for r is the \mathbb{RP}^{k-1} at distance $\pi/2$ from p , all invariant under W . In particular, $B := G \Lambda \subset M$ is a polar space form G submanifold of M with section $\Lambda = \mathbb{RP}^{k-1}$ and polar group induced from W . Indeed, $\langle r \rangle$ is central in W and $W/\langle r \rangle$ is the polar group of Λ and its lift of reflections to $\tilde{\Lambda} \subset \tilde{\Sigma}$ is $(\tilde{W}/\langle \tilde{r} \rangle) = W$. Arguing as above M is the union of a ball centered at p and a tubular neighborhood of B . (In the chosen metric B is the cut locus of p and vice versa at distance $\pi/2$). In particular, we have an equivariant sphere fiber bundle $p : \mathbb{S} \rightarrow B$ (with nontrivial fiber) between polar G manifolds with the same orbit space Δ_0 , where \mathbb{S} is the unit sphere at p . By transversality it is also clear that B is simply connected. Since, the assumptions about isotropy representations for p in the Hopf Lemma 7.1 above are clearly satisfied in our case, this completes the proof. \square

8. FIXED POINT FREE REDUCIBLE ACTIONS

In all remaining cases, the orbit space $M^* = \Sigma^*$ is a simplex Δ isometric to all chambers in M . Moreover, Δ is a spherical join $\Delta = \Delta_- * \Delta_+ = \Delta^{m_-} * \Delta^{m_+}$, corresponding to two dual W invariant subsections Σ_- and Σ_+ , where $\Sigma_{\pm} = \mathbb{S}^{m_{\pm}}$ or the projective spaces $\mathbb{RP}^{m_{\pm}}$.

Viewing Δ also as a subset of a fixed section Σ , clearly $B_- = G \Delta^{m_-}$ and $B_+ = G \Delta^{m_+}$ are two polar G submanifolds in M with sections $\Sigma_-, \Sigma_+ \subset \Sigma$ and Weyl groups $W \pmod{\text{kernel}}$. In particular, B_{\pm} are polar space forms of spherical type. Moreover, just like Δ can be viewed as the union of two tubular neighborhoods of the $\Delta^{m_{\pm}}$, M is the union of tubular neighborhoods of the G submanifolds B_{\pm} .

Note, that when say $m_+ \geq 1$, the slice representation at each vertex of $\Delta_+ \subset \Delta$ is reducible. In particular, all vertex representations are reducible except possibly the one corresponding to say Δ_- when it is a point.

In this section we consider the fixed point free case. Our goal in this section is to prove the following

THEOREM 8.1 (Non-fixed point). *A reducible fixed point free polar action on a simply connected positively curved manifold is equivariantly diffeomorphic to an isometric polar action on a rank one symmetric space, excluding the Cayley plane.*

The following is a key step is based on the primitivity lemma 3.2

LEMMA 8.2 (Dual Generation). *For any regular pair $p_{\pm} \in B_{\pm}$, the action of the isotropy groups $G_{p_{\pm}}$ restricted to B_{\mp} is orbit equivalent to the action of G restricted to B_{\mp} .*

Proof. By the primitivity theorem, G_{p_-} is generated by the face isotropy groups, $G_{v_1}, \dots, G_{v_{m_++1}}$ of the faces, $\Delta_- * \Delta_{v_i}^{m_+-1}$ containing Δ_- , and similarly G_{p_+} is generated by the remaining face isotropy groups, $G_{u_1}, \dots, G_{u_{m_-+1}}$, namely of the faces, $\Delta_{u_j}^{m_- - 1} * \Delta_+$ containing Δ_+ . Note that any face containing Δ_- is perpendicular to any face containing Δ_+ . In particular, if $G_{v,u}$ is the isotropy group at an intersection point of two such faces with isotropy groups G_v and G_u , the slice representation of $G_{v,u}$ restricted to the normal sphere of its fixed point set is a reducible cohomogeneity one action with singular isotropy groups G_v and G_u . As a special case of the

primitivity theorem we already know that G_v and G_u generate $G_{v,u}$. However, since the action is reducible we have that actually $G_v G_u = G_u G_v = G_{v,u}$ as sets. Notice that this is equivalent to that in the slice representation of $G_{v,u}$, the isotropy group G_v is transitive on the opposite singular orbit and vice versa. This is immediate for the standard reducible actions. For the exceptional cases (cf. Table 9.2), one easily checks the claim from the list of the seven cases.

We now claim that $G = G_{p_-} G_{p_+}$. From the primitivity lemma we know that any $g \in G$ can be written as a word of elements from $G_{v_1}, \dots, G_{v_{m_++1}}, G_{u_1}, \dots, G_{u_{m_-+1}}$. Using that $G_{v_i} G_{u_j} = G_{u_j} G_{v_i}$ for all $i = 1, \dots, m_++1$ and $j = 1, \dots, m_-+1$ we can rewrite any such word also as a word in the G_v 's times a word in the G_u 's, i.e., $G = G_{p_-} G_{p_+}$. The same reasoning shows that $G = G_{p_+} G_{p_-}$, and hence completes the proof of the lemma. \square

The above lemma will allow us to use the input from the previous section. For this we let $\Gamma(p_\pm) \subset M$ be the set consisting of all points on the set of all minimal geodesics from regular points p_\pm to B_\mp . Note that $\Gamma(p_-) \cap \Gamma(p_+)$ is the set of all minimal geodesics joining p_- and p_+ . We notice that $\Gamma(p_\pm)$ is the union of points in all the closed chambers with p_\pm as a common vertex, i.e. in the residue of p_\pm . Since G_{p_\pm} is independent of p_\pm , we will use the notation G_\pm instead.

It will also be useful to let $\Gamma(p_\pm)(r)$ denote the subset of $\Gamma(p_\pm)$ at distance r from p_\pm , and to let $\hat{\Gamma}(p_\pm)$ denote the negative of the terminal directions of the geodesics in $\Gamma(p_\pm)$.

Remark 8.3. The following are immediate consequences of the Dual Generation lemma 8.2, and the decomposition of a section $\Sigma \supset \Sigma_\pm$ corresponding to $\Delta = \Delta_- * \Delta_+$.

- The cut locus $C(B_\pm) = B_\mp$, and B_\pm are at distance $\pi/2$ from one another.
- $\Gamma(p_\pm) - B_\mp$ is a smooth submanifold of M diffeomorphic to the open $\pi/2$ ball in T_\pm^\perp via the exponential map
- The map $\gamma_{p_\pm} : \mathbb{S}_{p_\pm}^\perp \rightarrow B_\mp$ taking a unit vector to the corresponding geodesic at time $\pi/2$ is smooth, G_\pm equivariant and takes chambers to chambers.

We claim that the map γ_{p_\pm} is a G_\pm equivariant Hopf fibration. For the sake of simplicity we will use \bar{x} to denote the image $\gamma_{p_\pm}(x)$ of $x \in \mathbb{S}_{p_\pm}^\perp$. By the Hopf lemma 7.1 it suffices to verify that the slice representations of $G_{\pm,v}$ and $G_{\pm,\bar{v}}$ are orbit equivalent. We notice that, if x is a point of principal orbit type of the slice representation of $G_{\pm,v}$, we claim that $G_{\pm,v}(x) = G_{\pm,\bar{v}}(\bar{x})$. By the Dual Generation lemma 8.2 we clearly have that $G_{\pm,\bar{v}} = G_{\pm,\bar{x}} G_{\pm,v} = G_{\pm,v} G_{\pm,\bar{x}}$. Therefore, the orbit $G_{\pm,\bar{v}}(\bar{x}) = G_{\pm,\bar{v}} / G_{\pm,\bar{x}} = G_{\pm,v} G_{\pm,\bar{x}} / G_{\pm,\bar{x}}$. In particular, $G_{\pm,v}$ acts transitively on the orbit with isotropy group $G_{\pm,v} \cap G_{\pm,\bar{x}}$. It is clear that $G_{\pm,v} \cap G_{\pm,\bar{x}} = G_{\pm,x}$. The desired claim follows.

We are now ready to prove

LEMMA 8.4 (Reduction). *For all regular $p_\pm \in B_\pm$, $\Gamma(p_\pm)$ are G_\pm invariant submanifolds of M . Moreover,*

- $\Gamma(p_\pm)$ is G_\pm equivariantly diffeomorphic to $\mathbb{D}_{p_\pm}^\perp$ if the section is a sphere, and
- $\Gamma(p_\pm)$ are all G_\pm equivariantly diffeomorphic to a complex or quaternionic projective space if the section is a projective space.

In particular, B_\pm are chamber system connected polar space forms.

Proof. The key issue is to see that $\Gamma(p_{\pm})$ are submanifolds as claimed. From the remark above this is clear except along $B_{\mp} \subset \Gamma(p_{\pm})$.

From the Hopf lemma 7.1 and the above remark we know that $\gamma_{p_{\pm}} : \mathbb{S}_{p_{\pm}}^{\perp} \rightarrow B_{\mp}$ is either a diffeomorphism or a Hopf map. Clearly, $\Gamma_{p_{-}} \cap \Gamma_{p_{+}}$ is in bijective correspondence with the fiber of $\gamma_{p_{-}}$ over p_{+} and the fiber of $\gamma_{p_{+}}$ over p_{-} . In particular, both maps are of the same type, corresponding to $\Gamma_{p_{-}} \cap \Gamma_{p_{+}}$ being either one geodesic, an \mathbb{S}^1 , an \mathbb{S}^3 or an \mathbb{S}^7 family of geodesics.

Now consider the initial vectors of the geodesics in $\Gamma_{p_{-}}$ starting at B_{+} . This subset $\hat{\Gamma}(p_{-})$ of the unit normal bundle $T_1^{\perp} B_{+}$ of B_{+} is canonically a smooth submanifold diffeomorphic to $\mathbb{S}_{p_{-}}^{\perp}$ via $\Gamma(p_{-})(\frac{\pi}{2} - 1)$.

If the section is a real projective space, $\Gamma_{p_{-}} \cap \Gamma_{p_{+}}$ contains at least two geodesics, so $\gamma_{p_{\pm}}$ are Hopf fibrations, with fibers of dimension 1 or 3: Since by assumption neither B_{\pm} is a point, the presence of a 7-sphere fiber would imply that both B_{\pm} would be homotopy 8-spheres, and M a 24-dimensional manifold with non-trivial homology only at $8i$, for $i = 0, 1, 2, 3$. A contradiction by well-known topological theorem (cf. [Ha]).

Using this for $\gamma_{p_{-}}$ it follows that the initial vectors of $\Gamma_{p_{-}} \cap \Gamma_{p_{+}}$ at p_{-} is a linear 1- or 3- sphere in $\mathbb{S}_{p_{-}}^{\perp}$. Similarly, the initial vectors of $\Gamma_{p_{-}} \cap \Gamma_{p_{+}}$ at p_{+} is a linear 1- or 3- sphere in $\mathbb{S}_{p_{+}}^{\perp}$. It is not difficult to see that the same conclusion holds for all p_{\pm} along Δ_{\pm} , and hence by invariance everywhere. This shows that $\hat{\Gamma}(p_{-})$ is a linear \mathbb{S}^1 or \mathbb{S}^3 subbundle of the unit normal bundle of B_{+} . It follows that $\Gamma_{p_{-}}$ is a G_{-} invariant polar submanifold of M equivariantly equivalent to a polar action on a projective space. The same reasoning applies to $(\Gamma_{p_{+}}, G_{+})$.

If the section is a sphere, it follows from the above that $\gamma_{p_{\pm}}$ are diffeomorphism, and hence $\hat{\Gamma}(p_{\pm})$ smooth sections of the normal bundles of B_{\mp} . This completes the proof. \square

Having dealt with all cases where the Diagram for the Coxeter matrix has no isolated nodes, and where the action has fixed points, we assume from now on that B_{-} is an orbit corresponding to an isolated node of the diagram. Thus, we will use a decomposition

$$\Delta = \Delta_{-} * \Delta_{+}, \text{ where } \Delta_{-} = \Delta^0 = p_{-} \text{ is a vertex,}$$

corresponding to an isolated node, and Δ_{+} corresponds to the rest of the diagram. In particular,

$$G_{+} \text{ acts transitively on } B_{-}, \text{ as well as on each normal sphere } \mathbb{S}_{+}^{\perp} \text{ to } B_{+} \text{ along } \Delta_{+}.$$

We also need the following

LEMMA 8.5 (linearity). *Let (K, \mathbb{S}^n) be a polar representation, whose type M contains no isolated nodes. If an extended spherical polar action by $L \supset K$ is orbit equivalent to the K action, then it is linear as well.*

Proof. The case of cohomogeneity one follows directly from the classification by Straume [St]. In cohomogeneity at least 2, both actions have the same chamber system, which since K is linear, is a topological building. From [BSp], [GKMW] we know that the topological automorphism group Aut_{top} of the building is a Lie group, whose maximal compact sub groups acts linearly on the sphere orbit equivalent to Aut_{top} . Since both $K \subset L$ are compact subgroups of Aut_{top} the desired result follows. \square

Proof of Theorem 8.1. We first consider the case where the section $\Sigma = \Sigma_{-} * \Sigma_{+}$ is a sphere: By Lemma 8.4 the G_{\mp} action on B_{\pm} is equivariantly equivalent to the slice representation on the normal sphere $\mathbb{S}_{p_{\mp}}^{\perp}$ which we will denote by $\mathbb{S}(V_{\pm}) = \mathbb{S}_{\pm}$. Note that G as well as G_{+}

acts transitively on B_- since Δ_- is a point. In particular G acts linearly on \mathbb{S}_- identified with B_- . If also the G action on B_+ when identified with \mathbb{S}_+ is linear, we claim that the induced sum action on $\mathbb{S}(V_- \oplus V_+)$ is equivalently diffeomorphic to the G action on M . To see this, choose $p_-^* \in \mathbb{S}_-$ with $G_{p_-^*} = G_{p_-} = G_-$ and a G_- equivariant diffeomorphism from $\Gamma(p_-)$ to the join $p_-^* * \mathbb{S}_+ \subset \mathbb{S}_- * \mathbb{S}_+$. This extends to a well defined G equivariant diffeomorphism from M to $\mathbb{S}_- * \mathbb{S}_+$ by invariance. By the above lemma this completes the proof when Coxeter matrix for the G_- action on B_+ when identified with \mathbb{S}_+ has no isolated nodes. By the argument just given the general case follows by induction on the number of isolated nodes.

Now suppose Σ is a projective space:

By Lemma 8.4 the Cayley plane can not appear. Therefore we may assume that B_{\pm} are both projective spaces over \mathbb{C} or \mathbb{H} . Notice that, by Table 9.2, none of the reducible representation of exceptional type, considered as a linear action on the sphere, descend to an action on a complex or quaternionic projective space. In particular, the slice representations of all vertices in Δ_+ are either standard or special.

Let $K_{\pm} \triangleleft G$ be the identity connected component of kernel of the G -action on B_{\pm} . We divide the proof into two cases corresponding to (a) $\dim B_- > 2$ and (b) $\dim B_- = 2$ (noting that $\dim B_- < 2$ is covered by the fixed point case).

In case (a), in fact B_- is a projective space of real dimension at least 4. We claim that in this case K_+ acts transitively on all normal spheres to B_+ , hence also transitively on B_- . As a consequence, the kernel of the K_+ action on B_- , say K_0 , is $1, S^1$ or S^3 . To prove the claim, observe for dimension reasons that, for each vertex $v_i \in \Delta_+$, there is a normal simple subgroup $H_{v_i} \triangleleft G_{v_i}$ which acts transitively on the normal sphere to B_+ at v_i and trivially on the slice tangent to B_+ . In particular, $H_{v_i} \triangleleft G_+$, the principal isotropy group of B_+ . If v_j is another vertex of Δ_+ different from v_i , we get similarly $H_{v_j} \triangleleft G_+$. It is clear that $H_{v_i} = H_{v_j} \triangleleft G_+$ is the simple group factor of G_+ dividing the kernel. Here there is no ambiguity because the dimension assumption. By the primitivity $G = \langle G_{v_i}, G_{v_j} \rangle$, hence $H_+ = H_{v_i} = H_{v_j}$ is normal in G . Therefore, it fixes B_+ , i.e., $H_+ \triangleleft K_+ \triangleleft G$, and our claim follows. Hence, $K_+ = H_+ \cdot K_0$.

Let us write $G = K_+ \cdot L$. Then L acts in a polar fashion on B_+ . Moreover, from the list of transitive actions on projective spaces, and the fact that both $K_+ \cdot L = H_+ \cdot K_0 \cdot L$ and K_+ act transitively on B_- it follows that $K_- = L \cdot K_0$. In particular, the L action on $\mathbb{S}_{p_-}^{\perp}$ descends to the L action on B_+ , which in turn is orbit equivalent to the G action on B_+ . Therefore, the (possibly non polar) linear sum action by $G/K_0 = H_+ \cdot L$ on $\mathbb{S}(V_- \oplus V_+)$, where $\mathbb{S}(V_{\pm}) := \mathbb{S}_{p_{\pm}}^{\perp}$ descends to a (possibly non-polar) action on the quotient projective space $\mathbb{P}(V_- \oplus V_+)$. Note also that from the slice representations along B_+ it follows that the G -action on M is orbit equivalent to the subaction by $H_+ \cdot L$, because H_+ is transitive on the normal sphere to B_+ .

We claim, that $\mathbb{P}(V_- \oplus V_+)$ is $H_+ \cdot L$ equivariantly diffeomorphic to M . To see this, choose a point $\bar{p}_- \in \mathbb{P}(V_-)$ so that the isotropy group $(H_+)_{\bar{p}_-} = (H_+)_{p_-}$. By Lemma 8.4 we fix an L -equivariant diffeomorphism $F : \Gamma(p_-) \rightarrow \mathbb{P}(V_+ \oplus \mathbb{C}\bar{p}_-)$ resp. $\mathbb{P}(V_+ \oplus \mathbb{H}\bar{p}_-)$, where $\tilde{p}_- \in V_-$ has image $\bar{p}_- \in \mathbb{P}(V_-)$. We now extend F using H_+ -invariance to obtain an H_+ -diffeomorphism from M to $\mathbb{P}(V_- \oplus V_+)$. Specifically, for any $x \in \Gamma(p_-)$, where $p'_- = g(p_-)$ for some $g \in H_+$, we let $F(x) := gF(g^{-1}x)$. Since L and H_+ commute, this is well-defined and yields the desired $H_+ \cdot L$ diffeomorphism.

To complete the proof in this case we need to see that the induced G action on $\mathbb{P}(V_- \oplus V_+)$ is indeed polar relative to the standard structure.

When $\mathbb{P}(V_- \oplus V_+)$ is a complex projective space it follows that $K_0 \subset S^1$. In particular, if K_0 is non-trivial both linear H_+ and L actions extend to polar actions on \mathbb{S}_\mp descending to $\mathbb{P}(V_\mp)$. Notice, that the extended G action on $\mathbb{P}(V_- \oplus V_+)$ via M is indeed polar (weighted), because the subaction of $H_+ \cdot L$ is polar.

When $\mathbb{P}(V_- \oplus V_+)$ is a quaternionic projective space, we claim that K_0 is trivial. Indeed, if B_- is a quaternionic projective space, then the normal G equivariant neighborhood $G \times_{G_-} V_-$ of the orbit B_- is linear equivalent to the direct sum of the nontrivial Hopf \mathbb{H} -bundles by the above claim. In fact it can be identified with the free $\mathrm{Sp}(1)$ -quotient $\mathbb{S}(V_+) \times_{\mathrm{Sp}(1)} V_-$ as an H_+ -bundle, where $\mathrm{Sp}(1)$ acts diagonally and freely on $\mathbb{S}(V_\pm)$. Notice, however, K_0 acts as a subaction on the Hopf action on $\mathbb{S}(V_-)$ (hence coincide with the $\mathrm{Sp}(1)$ -action on $\mathbb{S}(V_-)$), but acts trivially on the tangential direction since it fixes B_- . This implies that the subaction of K_0 is trivial, since $\Delta(\mathrm{Sp}(1))$ does not commute with $1 \times \mathrm{Sp}(1)$ in $\mathrm{Sp}(1) \mathrm{Sp}(1)$.

Finally, let us consider the only remaining case (b) where $B_- = \mathbb{CP}^1$ and B_+ is a complex projective space of real dimension at least 4. Let us write, $G = K_- \cdot S^3$, where K_- is the kernel of the G action on B_- . Notice that $G_- = K_- \cdot S^1$, where $S^1 \subset S^3$. We claim that

- The factor S^3 acts trivially on B_+ .

For proving this let us write $\Delta_+ = \Delta_1 * \cdots * \Delta_k$, a decomposition of the orbit space such that every factor has a connected subdiagram. Let $\pi : B_+ \rightarrow \Delta_+$ be the orbit projection. It is easy to see that $\pi^{-1}(\Delta_i)$, for each i , is a complex projective space. The desired claim is clearly true for the subaction on $\pi^{-1}(\Delta_i)$, if the Coxeter diagram has only a node, by the list of transitive actions on a projective space. If the diagram contains more than 2 nodes, we see immediately from the main result in section 4 that both $K_- \cdot S^1$ and $K_- \cdot S^3$ actions on $\pi^{-1}(\Delta_i)$ lift to orbit equivalent linear actions on a sphere, and the same is true if the connected diagram contains two nodes by [Uc] and [EH]. In general the claim follows inductively, since, if S^3 acts trivially on both $\pi^{-1}(\Delta_i)$ and $\pi^{-1}(\Delta_j)$, it acts on $\Gamma(p_i)$ and also on $\Gamma(p_j)$ with complex codimension one fixed point set, considered as submanifolds in $\pi^{-1}(\Delta_i * \Delta_j)$. This clearly implies the S^3 -action is trivial on $\pi^{-1}(\Delta_i * \Delta_j)$ (the slice representation of S^3 of complex dimension 1 is trivial).

Therefore, K_- acts on B_+ in a polar fashion. We are now exactly in the same situation as above, and hence M is G -equivariantly diffeomorphic to $\mathbb{P}(\mathbb{C}^2 \oplus V_+)$. The desired result follows. \square

We point out that it follows from the above proof that exceptional reducible isotropy representations for a polar action in positive curvature only occur for polar actions on the spheres.

9. APPENDIX: EXCEPTIONAL REDUCIBLE POLAR REPRESENTATIONS

In this appendix we bring a classification of reducible polar representations that is a slightly corrected version of [Be].

We first consider reducible representations with two irreducible nontrivial summands, i.e., we let a connected compact Lie group G act on $V_+ \oplus V_-$, where each V_\pm is nontrivial, G invariant and irreducible. We let G_+ denote the identity component of the kernel of the action on V_- , and define G_- similarly. Then each G_\pm is a normal subgroup of G , and there is a normal subgroup K of G such that $G = G_+ \cdot G_- \cdot K$. With this notation, we make the following definitions.

- *Standard*: The action of G is orbit equivalent to the action of $G_+ \cdot G_-$.

- *Special*: The action of G is not standard, and the actions of G and G_i on V_i are orbit equivalent either for $i = +$ or for $-$ (but not for both).
- *Exceptional*: The action of G is not standard, and the actions of G and G_i on V_i are not orbit equivalent for both $i = +$ and $-$.

Remark 9.1. (i) If the action of G is standard, then it follows from [EH] that K is either S^1 or S^3 if it is nontrivial. An example of such an action with nontrivial K is $G = S^1 \cdot SU(n) \cdot SU(m)$ acting on $\mathbb{C}^n \oplus \mathbb{C}^m$ by $(\lambda, A, B)(v, w) = (\lambda Aw, \lambda Bw)$. Here $K = S^1$.

(ii) If the action of G is special, then K is nontrivial and it is either S^1 or S^3 ; see Theorem 9.2. It is possible that both G_+ and G_- are nontrivial. An example of this is the action of $S^1 \cdot SU(n) \cdot SO(k)$ on $\mathbb{C}^n \oplus \mathbb{C}^k$ defined by $(\lambda, A, B)(w, v) = (\lambda A(w), \lambda v B')$. It is also possible that either G_+ or G_- is trivial. An example of a special action where G_- is trivial is $U(n)$ acting on $\mathbb{C}^n \oplus \mathbb{C}$ by $A(w, z) = (Aw, \det(A)z)$.

(iii) There is a complete list of exceptional actions in Table 9.2. They are of two rather different types. In the first four examples in the table, the groups G_+ and G_- are trivial, i.e., $G = K$. The cohomogeneity of the actions on $V_+ \oplus V_-$ is two in all four cases. In the last three examples in the table $G_+ = SO(2)$ or $SO(3)$, G_- is trivial, and $K = Spin(7)$ or $Spin(8)$. The cohomogeneity of the three actions on $V_+ \oplus V_-$ is three or four.

The following theorem is in part based on results of Bergmann in [Be] where the exceptional reducible representations are referred to as *interesting*.

THEOREM 9.2. *Let $\rho : G \rightarrow SO(V_+ \oplus V_-)$ be an almost effective reducible polar representation where V_+ and V_- are both irreducible. We assume that ρ is not standard. Then either V_+/G or V_-/G is one-dimensional. We assume that V_+/G is one-dimensional.*

- (i) *If ρ is special, we have the following two possibilities:*
- *G_- is trivial and $V_+ \oplus V_- = \mathbb{C}^n \oplus \mathbb{C}, \mathbb{H}^n \oplus \mathbb{C}, \mathbb{H}^n \oplus \mathbb{R}^3$ or $\mathbb{H}^n \oplus \mathbb{H}$, where $G = U(n), Sp(n)U(1)$ or $Sp(n)Sp(1)$. Here V_-/G is one-dimensional.*
 - *Both G_+ and G_- are non-trivial, $G_+ \cdot K$ is transitive on the unit sphere in V_+ ($= \mathbb{C}^n$ or \mathbb{H}^n) with $G_+ = SU(n)$ or $Sp(n)$ acting in a standard fashion, where $K = S^1$ or $Sp(1)$, $n \geq 2$. The action of G on V_- is orbit equivalent to a polar representation of $S^1 \cdot G_-$ or $Sp(1) \cdot G_-$ on V_- . Here V_-/G is at least two-dimensional.*
- (ii) *If ρ is exceptional, it is one of the representations in Table 9.2. In the first four examples in the table, V_-/G is one-dimensional, and in the last three it is two- or three-dimensional.*

We first prove a lemma.

LEMMA 9.3. *Let $\rho : G \rightarrow SO(V_+ \oplus V_-)$ be an almost effective reducible polar representation where V_+ and V_- are both irreducible. Then we have one of the following two possibilities.*

- (i) *The representation ρ is standard.*
- (ii) *Either V_+/G or V_-/G is one-dimensional.*

Proof. As before, we let G_- denote the kernel of the G action on V_+ . Then $G = L_+ \cdot G_-$, where L_+ is the almost effective part of the G -action on V_+ . If the action of L_+ is trivial on V_- , then

TABLE 9.2. Exceptional reducible polar repr. of compact connected Lie groups

Type	Group G	Repr. space	Principal orbits	Isotropy group K_1	Isotropy group K_2
A_3	$SU(4) \cong \text{Spin}(6)$	$\mathbb{R}^6 \oplus \mathbb{R}^8$	$\mathbb{S}^5 \times \mathbb{S}^7$	$SU(3)$	$\text{Spin}(5) \cong \text{Sp}(2)$
$A_3 \times S^1$	$U(4)$	$\mathbb{R}^6 \oplus \mathbb{R}^8$	$\mathbb{S}^5 \times \mathbb{S}^7$	$U(3)$	$\text{Sp}(2) \cdot S^1$
B_3	$\text{Spin}(7)$	$\mathbb{R}^7 \oplus \mathbb{R}^8$	$\mathbb{S}^6 \times \mathbb{S}^7$	G_2	$\text{Spin}(6) \cong SU(4)$
D_4	$\text{Spin}(8)$	$\mathbb{R}^8 \oplus \mathbb{R}^8$	$\mathbb{S}^7 \times \mathbb{S}^7$	$\text{Spin}(7)$	$\text{Spin}(7)$
$B_3 \times S^1$	$\text{Spin}(7) \times SO(2)$	$\mathbb{R}^8 \oplus \mathbb{R}^{14}$	$\mathbb{S}^7 \times \mathbb{M}^{12}$	$G_2 \times SO(2)$	$\text{Spin}(5) \cong \text{Sp}(2)$
$D_4 \times S^1$	$\text{Spin}(8) \times SO(2)$	$\mathbb{R}^8 \oplus \mathbb{R}^{16}$	$\mathbb{S}^7 \times \mathbb{M}^{14}$	$\text{Spin}(7) \times SO(2)$	$\text{Spin}(6) \cong SU(4)$
$D_4 \times A^1$	$\text{Spin}(8) \times SO(3)$	$\mathbb{R}^8 \oplus \mathbb{R}^{24}$	$\mathbb{S}^7 \times \mathbb{M}^{21}$	$\text{Spin}(7) \times SO(3)$	$\text{Spin}(5) \cong \text{Sp}(2)$

G_- on V_- is orbit equivalent to the G -action, and we set $G_+ = L_+$. Then the action of $G_+ \cdot G_-$ is orbit equivalent to the action of G , proving the claim in (i) of the lemma.

Notice that a principal isotropy group at V_+ , i.e., $(L_+)_p \cdot G_-$ and G are orbit equivalent on V_- , and $(L_+)_p$ is a proper subgroup of L_+ .

Let us assume that the action of L_+ on V_- is not trivial. Then by [EH] we know that, $G_- = \{1\}$, S^1 or $\text{Sp}(1)$. If V_-/G is not one-dimensional, it follows from [EH] that $(L_+ \cdot G_-)/(L_+)_p \cdot G_-$ is \mathbb{S}^1 or \mathbb{S}^7 . This implies that the principal orbits in V_+ are one- or seven-dimensional spheres and hence that V_+/G is one-dimensional, and we are in the situation in (ii) in the claim of the lemma. This finishes the proof of the lemma. \square

Proof of Theorem 9.2. By Lemma 9.3, if both V_+/G and V_-/G are at least two-dimensional, then ρ is standard. We can therefore assume that V_+/G is one-dimensional.

Let $G = L_+ \cdot G_-$ be as in the proof of Lemma 9.3. We can assume that L_+ does not act trivially on V_- . If V_-/G is at least two-dimensional, we have the following two possibilities:

- The action of L_+ on V_- is almost effective. Then by [EH], the pair $(L_+, (L_+)_p)$ is $(\text{Spin}(7), G_2)$ or $(\text{Spin}(8), \text{Spin}(7))$, with $G_- = S^1$ or $\text{Sp}(1)$. The representation ρ is then one of the last three representations in Table 9.2. (Their types are $B_3 \times S^1$, $D_4 \times S^1$, and $D_4 \times A_1$.) These representations are exceptional.

- The action of L_+ on V_- has a kernel of positive dimension. Then L_+ is not a simple group, and it acts transitively on a sphere. The pair $(L_+, (L_+)_p)$ is therefore $(U(n), U(n-1))$, $(\text{Sp}(n) \text{Sp}(1), \text{Sp}(n-1) \text{Sp}(1))$, or $(\text{Sp}(n) U(1), \text{Sp}(n-1) U(1))$ where $n \geq 1$. The kernel is clearly $SU(n)$ or $\text{Sp}(n)$. Hence the action of G on V_- is orbit equivalent to the action of $\det(U(n)) \cdot G_-$ or $\text{Sp}(1) \cdot G_-$ on V_- . Since it is an irreducible polar representation, the classifications in [Da] and [EH] together with the description of isotropy representations of symmetric spaces in [Bes], pp. 312–314, give us all possibilities for G_- and the action of $\det(U(n)) \cdot G_-$ or $\text{Sp}(1) \cdot G_-$ on V_- .

It remains only to consider the case both V_+/G and V_-/G are one-dimensional. Such representations are listed in Appendix II on pp. 102–107 in [GWZ]. There are first the so-called essential cohomogeneity two representations (which restrict to cohomogeneity one actions on spheres) which are listed in Tables E and F on pp. 105–106 in [GWZ]. The reducible essential cohomogeneity two representations turn out to be exactly the first four cases in Table 9.2. Now let us assume that the reducible cohomogeneity two representation is not essential. This case is discussed on pp. 105–106 in [GWZ]. Assuming that the representation is also not standard, we see that $G = U(n)$, $\text{Sp}(n) U(1)$ and $\text{Sp}(n) \text{Sp}(1)$ acting transitively on $\mathbb{C}^n \oplus \mathbb{C}$, $\mathbb{H}^n \oplus \mathbb{C}$, $\mathbb{H}^n \oplus \mathbb{R}^3$,

or $\mathbb{H}^n \oplus \mathbb{H}$. These representations are special of the second kind. This finishes the proof of the theorem. \square

Now we turn to the case that the reducible polar action contains at least three irreducible components. We conclude this section with the following lemma which will be used in the proofs of Theorem 7.1 and Theorem 8.1.

LEMMA 9.4. *Let $\rho : \mathbf{G} \rightarrow \mathrm{SO}(V_1 \oplus \cdots \oplus V_k)$ be a reducible polar representation where $k \geq 3$ and the V_i are irreducible \mathbf{G} -modules. Assume that none of the $\rho_{ij} : \mathbf{G} \rightarrow \mathrm{SO}(V_i \oplus V_j)$ is exceptional for some $i \neq j$. If V_1/\mathbf{G} is one-dimensional, then we have one of the following two possibilities.*

- (i) $\mathbf{G} = \mathbf{G}_1 \cdot \mathbf{G}_{2\dots k} \cdot \mathbf{K}$, where $\mathbf{K} = \{1\}, \mathbf{S}^1$ or $\mathrm{Sp}(1)$, the sub action of $\mathbf{G}_1 \cdot \mathbf{G}_{2\dots k}$ on $V_1 \oplus (V_2 \oplus \cdots \oplus V_k)$ is in a componentwise fashion. Moreover, the subaction is orbit equivalent to the \mathbf{G} -action, if and only if none of the ρ_{1j} are special for all $j \geq 2$.
- (ii) $V_1 = \mathbb{C}$ or \mathbb{R}^3, \mathbb{H} , $\mathbf{G} = \mathbf{G}_{2\dots k} \cdot \mathbf{K}$, where $\mathbf{K} = \mathbf{S}^1, \mathrm{Sp}(1)$ acts transitively on $\mathbb{S}(V_1)$, and $\mathbf{G}_{2\dots k}$ and \mathbf{G} are orbit equivalent on $V_2 \oplus \cdots \oplus V_k$.

Proof. As above we can write $\mathbf{G} = \mathbf{L}_1 \cdot \mathbf{G}_-$ where \mathbf{L}_1 is the almost effective part of the action of \mathbf{G} on V_1 and \mathbf{G}_- is as always the kernel of the action of \mathbf{G} on V_1 .

If $\dim(V_1) \geq 5$, we claim that there is a normal subgroup \mathbf{G}_1 of \mathbf{L}_1 acting transitively on $\mathbb{S}(V_1)$ but trivially on $V_2 \oplus \cdots \oplus V_k$. By Lemma 9.2(i) we get that, for any $j \geq 2$, there is a normal, simple subgroup $\mathbf{G}_{1j} \triangleleft \mathbf{L}_1$ which acts transitively on $\mathbb{S}(V_1)$ but trivially on V_j , thanks to the dimension assumption. The desired claim follows since $\mathbf{G}_{1,j} = \mathbf{G}_{1,j'}$ for all $2 \leq j, j' \leq k$. Let us write $\mathbf{G} = \mathbf{G}_1 \cdot \mathbf{K}_1$ where $\mathbf{K}_1 = \mathbf{G} / \mathbf{G}_1$. By the list of transitive action on the spheres we see immediately that \mathbf{K}_1 contains at most an \mathbf{S}^1 or an $\mathrm{Sp}(1)$ factor acting freely on $\mathbb{S}(V_1)$. Therefore, $\mathbf{K}_1 = \mathbf{G}_{2\dots k} \cdot \mathbf{K}$ where $\mathbf{G}_{2\dots k}$ is the kernel of the \mathbf{K}_1 action on V_1 , $\mathbf{K} = \{1\}, \mathbf{S}^1$ or $\mathrm{Sp}(1)$. It is clear that the action of $\mathbf{G}_1 \cdot \mathbf{G}_{2\dots k}$ is in a componentwise fashion. If ρ_{1j} is standard for all $j \geq 2$, then the factor $\mathbf{K} = \mathbf{S}^1$ or \mathbf{S}^3 is nonessential, and so the sub $\mathbf{G}_{2\dots k}$ action and the \mathbf{G} action are orbit equivalent on $V_2 \oplus \cdots \oplus V_k$.

Now let us assume $\dim(V_1) \leq 4$. Notice that \mathbf{L}_1 is then either \mathbf{S}^1 , $\mathrm{Sp}(1) = \mathrm{SO}(3)$, or $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$. Notice that the isotropy group $(\mathbf{L}_1)_p$ is either $\{1\}$, $\mathbf{S}^1 \subset \mathrm{SO}(3)$, or the diagonal subgroup $\Delta(\mathrm{Sp}(1)) \subset \mathrm{Sp}(1)\mathrm{Sp}(1)$, such that $(\mathbf{L}_1)_p \cdot \mathbf{G}_-$ acts on $V_2 \oplus \cdots \oplus V_k$ orbit equivalent to the \mathbf{G} -action (cf. [Be] Theorem 2). The restrictions of these orbit equivalent actions on the irreducible summand V_j , $j \geq 2$, which are either rank 1 or at least 2. By appealing to [EH] and the list of transitive actions on spheres, it is easy to see that $(\mathbf{L}_1)_p$ is *nonessential*, i.e, it can be dropped up to orbit equivalence, on every V_j for all $j \geq 2$, unless $\mathbf{L}_1 = \mathrm{SO}(4)$ acting on V_1 by the defining representation. In such a case $(\mathbf{L}_1)_p = \mathrm{SO}(3)$, we claim that there is at most one essential summand among V_j 's where the representation of $(\mathbf{L}_1)_p \cdot \mathbf{G}_-$ is the tensor representation of $\mathrm{SO}(3) \cdot \mathbf{G}_-$ (real tensor representation), or $\mathrm{SU}(2) \cdot \mathbf{G}_-$ (complex tensor representation). This follows inductively from Lemma 9.2, since, by [Be] Theorem 2, $\mathrm{SO}(3) \cdot \mathbf{G}_-$ is polar on $V_2 \oplus \cdots \oplus V_k$. Therefore, we let \mathbf{K} be either \mathbf{L}_1 or a factor of \mathbf{L}_1 when it is $\mathrm{Sp}(1)\mathrm{Sp}(1)$, let $\mathbf{G}_{2\dots k} = \mathbf{G} / \mathbf{K}$. It is easy to see that the $\mathbf{G}_{2\dots k}$ action and the \mathbf{G} action are orbit equivalent on $V_2 \oplus \cdots \oplus V_k$. The desired result follows. \square

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